MECH230 Section 2 - Fall 2024
Midterm 1 SOLUTIONS

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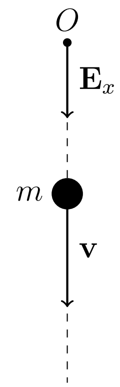
Saturday October 26, 2024 at 8:30-10:00 am

Instructions

- This is an online exam. You are allowed to consult your class notes, your problem set solutions, your homework, and the textbooks during the exam. However, you are not allowed to use AI tools such as ChatGPT. You are also not allowed to communicate with anyone during the exam regarding the exam.
- I reserve the right to question you orally about your solutions or to schedule an oral exam with you in the future.
- As per AUB policy, cheating will not be tolerated and will be penalized according to policy.
- This exam consists of 3 problems with several parts. Clearly indicate which part you are solving on your solution sheet.
- Read and follow the instructions carefully.
- Write in a legible and neat way. Illegible solutions will not be graded.
- Scan and submit your exam through the link provided in the email in which you received the exam. Please make sure that your scan is clear and that the pages are ordered and oriented correctly. As long as I receive your exam by 10:15 am, it will be accepted.
- I will post the midterm questions and solutions on the class website after the exam.

Problem 1

In a scientific experiment, a particle of mass m is moving in rectilinear motion in the $\mathbf{E}_x$ direction as shown in the figure. The only force acting on the object is $\mathbf{F}_D = -c\mathbf{v}$ where c is a constant and $\mathbf{v}$ is the velocity of the particle. No gravitational forces are acting on the particle. The origin is taken such that the position vector of the particle is $\mathbf{r} = x\mathbf{E}_x$.



(a) Differentiate the position vector of the particle to obtain its velocity and acceleration vectors.

$$\mathbf{v} = \dot{x}\mathbf{E}_x,$$
$$\mathbf{a} = \ddot{x}\mathbf{E}_x.$$

(b) Draw the free body diagram of the particle.

Draw $\mathbf{F}_D = -c\mathbf{v}$ pointing up, opposite to the velocity of the particle.

(c) Write the balance of linear momentum of the particle and solve for its acceleration.

$$\mathbf{F} = m\mathbf{a},$$
$$-c\dot{x}\mathbf{E}_x = m\ddot{x}\mathbf{E}_x,$$
$$-\frac{c}{m}\dot{x} = \ddot{x}.$$

(d) If the particle is initially located at the origin where it is given an initial speed is v_0, what is the distance traveled by the particle until it stops moving?

Using the relation $adx = vdv$, we can write

$$-\frac{c}{m}\dot{x}dx = \dot{x}d\dot{x},$$
$$-\frac{c}{m}dx = \dot{x}.$$

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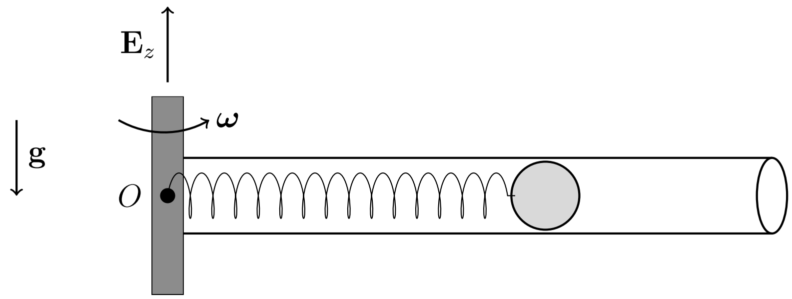
Integrating the above result between $\dot{x}_0 = v_0$ and $\dot{x}_{final} = 0$, we get

$$\Delta x = \frac{m}{c} v_0.$$

Problem 2

Consider a particle of mass m in a smooth tube that is rotating in the horizontal plane with angular velocity $\boldsymbol{\omega} = \dot{\theta}(t)\mathbf{E}_z$ where $\theta(t) = f(t)$. Inside the tube there is a spring of stiffness K and unstretched length ℓ_0 attached to the fixed end of the tube at the origin O at one end and to the particle at the other end as seen in the following figure.



(a) Copy the figure and draw the cylindrical-polar basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{E}_z\}$ on it. Also, write the expressions for the position, velocity, and acceleration vectors of the particle.

$$\mathbf{r} = r\mathbf{e}_r,$$
$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{f}(t)\mathbf{e}_\theta,$$
$$\mathbf{a} = \left(\ddot{r} - r(\dot{f}(t))^2\right)\mathbf{e}_r + \left(r\ddot{f}(t) + 2\dot{r}\dot{f}(t)\right)\mathbf{e}_\theta.$$

(b) Draw the free body diagram of the particle and write the expressions for the forces acting on it as applicable.

Draw a particle with the forces

$$\mathbf{W} = -mg\mathbf{E}_z,$$
$$\mathbf{N} = N_\theta\mathbf{e}_\theta + N_z\mathbf{E}_z,$$
$$\mathbf{F}_s = -k\left(r - \ell_0\right)\mathbf{e}_r.$$

acting on it.

(c) Write the vector equation of the balance of linear momentum of the particle.

$$\mathbf{F} = m\mathbf{a},$$
$$-mg\mathbf{E}_z + N_\theta\mathbf{e}_\theta + N_z\mathbf{E}_z - k(r-\ell_0)\mathbf{e}_r = m\left(\left(\ddot{r} - r(\dot{f}(t))^2\right)\mathbf{e}_r + \left(r\ddot{f}(t) + 2\dot{r}\dot{f}(t)\right)\mathbf{e}_\theta\right).$$

(d) Find the expression for the normal force applied by the tube to the particle.

$$(\cdot \mathbf{e}_\theta) \quad N_\theta = r\ddot{f}(t) + 2\dot{r}\dot{f}(t),$$
$$(\cdot \mathbf{E}_z) \quad N_z = mg,$$
$$\mathbf{N} = \left(r\ddot{f}(t) + 2\dot{r}\dot{f}(t)\right)\mathbf{e}_\theta + mg\mathbf{E}_z.$$

(e) Verify that the equation of motion of the particle can be written in the form

$$\ddot{r} + A(t)\,r + B = 0$$

where $A(t)$ is a function of time and B is a constant that you should determine.

$$m(\ddot{r} - r\dot{f}^2) + k(r - \ell_0) = 0,$$
$$\ddot{r} + \left(\frac{k}{m} - \dot{f}^2\right) r - \frac{k}{m}\ell_0 = 0.$$

(f) *(BONUS - No Partial Credit)* If the tube was rough with kinetic friction coefficient μ_k, what would the kinetic friction force acting on the particle be?

The relative velocity between the particle and the underlying point on the tube is

$$\mathbf{v}_{rel} = \mathbf{v}_{ball} - \mathbf{v}_{tube\ under\ ball} = (\dot{r}\mathbf{e}_r + r\dot{f}\mathbf{e}_\theta) - (r\dot{f}\mathbf{e}_\theta) = \dot{r}\mathbf{e}_r.$$

Hence,

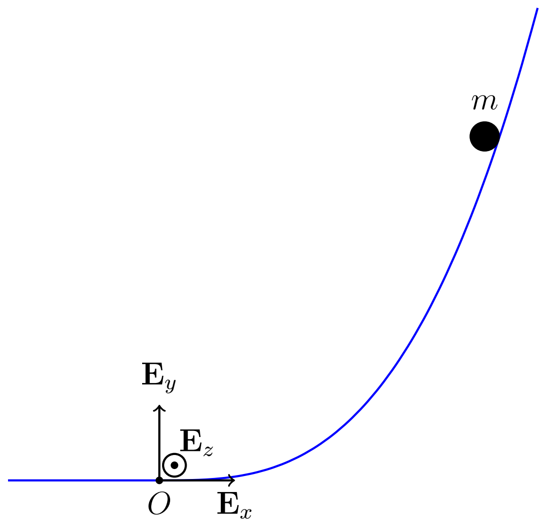
$$\mathbf{F}_f = -\mu_s||\mathbf{N}||\frac{\mathbf{v}_{rel}}{||\mathbf{v}_{rel}||} = -\mu_s m\sqrt{\left(r\ddot{f} + 2\dot{r}\dot{f}\right)^2 + g^2}\,\mathrm{sgn}(\dot{r})\mathbf{e}_r.$$

Problem 3

A child modeled as a particle of mass m is sliding down a slide with the equation $y = kx^3$ for $x \geq 0$ where k is a constant. So, the position vector from the origin to the slide is

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y = x\mathbf{E}_x + kx^3\mathbf{E}_y, \qquad 0 \leq x \leq x_{max}.$$

To the left of the origin, the curved slide is connected to a horizontal portion.



(a) Differentiate the given position vector to obtain the velocity and acceleration vectors of the particle.

$$\begin{aligned}
\mathbf{r} &= x\mathbf{E}_x + kx^3\mathbf{E}_y, \\
\mathbf{v} &= \dot{x}\left(\mathbf{E}_x + 3kx^2\mathbf{E}_y, \right) \\
\mathbf{a} &= \ddot{x}\mathbf{E}_x + 3kx\dot{x}\left(x + \dot{x}\right)\mathbf{E}_y.
\end{aligned}$$

(b) Express the unit tangent vector $\mathbf{e}_t$ in terms of the Cartesian basis vectors $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$. Use the fact that $\dot{x} \leq 0$ when m is going down the slide to simplify your answer.

$$\mathbf{e}_t = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{\dot{x}\left(\mathbf{E}_x + 3kx^2\mathbf{E}_y\right)}{|\dot{x}|\sqrt{1 + 9k^2x^4}} = -\frac{\mathbf{E}_x + 3kx^2\mathbf{E}_y}{\sqrt{1 + 9k^2x^4}}.$$

(c) Sketch the Serret-Frenet basis vectors $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ on several locations of the slide then express $\mathbf{e}_b$, the unit binormal vector, in the Cartesian basis.

$\mathbf{e}_t$ is tangent to the curve, $\mathbf{e}_n$ points toward the center of the curvature (leftward and upward), and $\mathbf{e}_b = -\mathbf{E}_z$.

(d) Use your answer from the previous part to deduce an expression for $\mathbf{e}_n$, the unit normal vector.

$$\mathbf{e}_n = \mathbf{e}_b \times \mathbf{e}_t = -\mathbf{E}_z \times \left(-\frac{\mathbf{E}_x + 3kx^2\mathbf{E}_y}{\sqrt{1+9k^2x^4}} \right) = \frac{\mathbf{E}_y - \mathbf{E}_x(3kx^2)}{\sqrt{1+9k^2x^4}}.$$

(e) Draw the free body diagram of the particle and write the expressions of the forces applied on it as applicable.

The forces acting on the particle are

$$\mathbf{N} = N\mathbf{e}_n,$$
$$\mathbf{W} = -mg\mathbf{E}_y.$$

(f) Write the vector equation of the balance of linear momentum of the particle.

$$\mathbf{F} = m\mathbf{a},$$
$$N\mathbf{e}_n - mg\mathbf{E}_y = m\left(\dot{v}\mathbf{e}_t + \kappa v^2\mathbf{e}_n\right).$$

(g) Solve for the normal force acting on the particle as it slides down the incline as a function of the curvature κ, x, $\dot{x}$ and other constant parameters.

$$N - \frac{mg}{\sqrt{1+9k^2x^4}} = \kappa v^2 = \kappa\dot{x}^2(1+9k^2x^4).$$
$$N = \kappa\dot{x}^2(1+9k^2x^4) + \frac{mg}{\sqrt{1+9k^2x^4}}.$$

(h) Comment on the safety of the child as he transitions from the curved slide to the horizontal portion at the left of origin.
Note: The cubic function $y = kx^3$ has an inflection point at $x = 0$ meaning that $\kappa = 0$ at $x = 0$.

The cubic has an inflection point at $x = 0$, so the curvature at the origin is zero which matches the curvature of the straight portion of the curve. Hence, the child will not experience a sudden change in acceleration as he transitions from the curved to the straight portion. Thus, the transition is safe.

MECH230 - Section 2
Midterm 1 Formula Sheet

Kinematics in Cartesian Coordinates

$$\begin{aligned}\mathbf{r} &= x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z,\\ \mathbf{v} &= v_x\mathbf{E}_x + v_y\mathbf{E}_y + v_z\mathbf{E}_z = \dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y + \dot{z}\mathbf{E}_z,\\ \mathbf{a} &= a_x\mathbf{E}_x + a_y\mathbf{E}_y + a_z\mathbf{E}_z = \ddot{x}\mathbf{E}_x + \ddot{y}\mathbf{E}_y + \ddot{z}\mathbf{E}_z.\end{aligned} \tag{1}$$

Rectilinear Motion Consider a rectilinear motion of a particle in the direction of $\mathbf{E}_x$.

$$\begin{aligned}\mathbf{r} &= x\mathbf{E}_x,\\ \mathbf{v} &= v\mathbf{E}_x = \dot{x}\mathbf{E}_x,\\ \mathbf{a} &= a\mathbf{E}_x = \ddot{x}\mathbf{E}_x.\end{aligned} \tag{2}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}. \tag{3}$$

Kinematics in Cylindrical Polar Coordinates

$$\begin{aligned}\mathbf{r} &= r\mathbf{e}_r + z\mathbf{E}_z,\\ \mathbf{v} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{E}_z,\\ \mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\mathbf{E}_z,\end{aligned} \tag{4}$$

where

$$\mathbf{e}_r = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \qquad \mathbf{e}_\theta = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y. \tag{5}$$

Kinematics in the Serret-Frenet Basis

$$v = ||\mathbf{v}|| = \frac{ds}{dt}, \quad \mathbf{e}_t = \frac{\mathbf{v}}{v}, \quad \frac{d\mathbf{e}_t}{ds} = \kappa\mathbf{e}_n, \quad \mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n, \quad \frac{d\mathbf{e}_b}{ds} = -\tau\mathbf{e}_n, \quad \rho = \frac{1}{\kappa}. \tag{6}$$

$$\begin{aligned}\mathbf{v} &= v\mathbf{e}_t.\\ \mathbf{a} &= \dot{v}\mathbf{e}_t + \kappa v^2\mathbf{e}_n.\end{aligned} \tag{7}$$

The Balance of Linear Momentum for a particle $\quad \mathbf{F} = \dot{\mathbf{G}}$ where $\mathbf{G} = m\mathbf{v}$.

Spring Forces A spring of stiffness K with unstretched length ℓ_0 whose base is at point A and whose free end is attached to a mass m with position vector $\mathbf{r}$ applies a force on m that is

$$\mathbf{F}_s = -K(||\mathbf{r} - \mathbf{r}_A|| - \ell_0)\frac{\mathbf{r} - \mathbf{r}_A}{||\mathbf{r} - \mathbf{r}_A||}. \tag{8}$$

Friction Forces

- Static friction is unknown but satisfies that static friction criterion $||\mathbf{F}_f|| \leq \mu_s||\mathbf{N}||$.
- Kinetic friction is prescribed according to Coulomb's friction model to be $\mathbf{F}_f = -\mu_k||\mathbf{N}||\frac{\mathbf{v}_{rel}}{||\mathbf{v}_{rel}||}$.