## MECH230 - Section 2 **DRAFT** Final Formula Sheet

Sine Law and Cosine Law



Referring to the above triangle with interior angles A, B, and C and with side lengths a, b, and c, the sine law and cosine law are respectively

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c},\tag{1}$$

$$c^{2} = a^{2} + b^{2} - 2ab\cos(C).$$
<sup>(2)</sup>

**Chain Rule** Let  $\theta$  be some function of time such that  $\boldsymbol{\omega} = \dot{\theta} \mathbf{E}_z$  and  $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \ddot{\theta} \mathbf{E}_z$ , then

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\theta}{dt}.$$
(3)

**Spring Force** A spring of stiffness K with unstretched length  $\ell_0$  whose base is at point A and whose free end is attached to a mass m with position vector **r** applies a force on m that is

$$\mathbf{F}_{s} = -K(||\mathbf{r} - \mathbf{r}_{A}|| - \ell_{0}) \frac{\mathbf{r} - \mathbf{r}_{A}}{||\mathbf{r} - \mathbf{r}_{A}||}.$$
(4)

## **Friction Force**

- Static friction is unknown but satisfies that static friction criterion  $||\mathbf{F}_{f}|| \leq \mu_{s} ||\mathbf{N}||$ .
- Kinetic friction is prescribed according to Coulomb's friction model to be  $\mathbf{F}_f = -\mu_k ||\mathbf{N}|| \frac{\mathbf{v}_{rel}}{||\mathbf{v}_{rel}||}$ .

Rigid Body (RB) Kinematics If A is fixed to the RB and B moving with respect to it, then

$$\mathbf{r}_{B} - \mathbf{r}_{A} = x\mathbf{e}_{x} + y\mathbf{e}_{y},$$

$$\mathbf{v}_{B} - \mathbf{v}_{A} = \boldsymbol{\omega} \times (\mathbf{r}_{B} - \mathbf{r}_{A}) + \mathbf{v}_{rel},$$

$$\mathbf{a}_{B} - \mathbf{a}_{A} = \boldsymbol{\alpha} \times (\mathbf{r}_{B} - \mathbf{r}_{A}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_{B} - \mathbf{r}_{A})) + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} + \mathbf{a}_{rel},$$
(5)

where the angular velocity and acceleration of the RB are resp.  $\boldsymbol{\omega} = \dot{\theta} \mathbf{E}_z$  and  $\boldsymbol{\alpha} = \ddot{\theta} \mathbf{E}_z$  and

 $\mathbf{v}_{rel} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y, \quad \text{and} \quad \mathbf{a}_{rel} = \ddot{x}\mathbf{e}_x + \ddot{y}\mathbf{e}_y. \tag{6}$ 

The corotation basis is taken such that

$$\mathbf{e}_x = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \quad \text{and} \quad \mathbf{e}_y = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y.$$
 (7)

Then,  $\dot{\mathbf{e}}_x = \boldsymbol{\omega} \times \mathbf{e}_x$  and  $\dot{\mathbf{e}}_y = \boldsymbol{\omega} \times \mathbf{e}_y$ .

Sum of Forces and Moments Consider a rigid body with K applied forces  $\mathbf{F}_i$  at points with position vectors  $\mathbf{r}_i$  and an applied couple  $\mathbf{M}^e$ , then the sum of forces and the moments about an arbitrary point P on the body are respectively

$$\mathbf{F} = \sum_{i=1}^{K} \mathbf{F}_{i}, \quad \text{and} \quad \mathbf{M}^{P} = \sum_{i=1}^{K} (\mathbf{r}_{i} - \mathbf{r}_{P}) \times \mathbf{F}_{i} + \mathbf{M}_{e}.$$
(8)

Balance of Linear Momentum (BoLM) The BoLM of a RB is written as

$$\mathbf{F} = \dot{\mathbf{G}} = m\mathbf{a}_C,\tag{9}$$

where  $\mathbf{G} = \int_{\mathcal{B}} \mathbf{v} dm = m \mathbf{v}_C$  is the linear momentum of the body  $\mathcal{B}$  and C is its center of mass,  $\mathbf{r}_C = \frac{\int_{\mathcal{B}} \mathbf{r} dm}{\int_{\mathcal{B}} dm}$ . C behaves like a material point of the rigid body.

The linear impulse - linear momentum equation is

$$\int_{t_A}^{t^B} \mathbf{F} dt = \mathbf{G}(t_B) - \mathbf{G}(t_A).$$
(10)

Balance of Angular Momenum (BoAM) The BoAM of a RB has three equivalent forms

$$\mathbf{M}^{O} = \dot{\mathbf{H}}^{O} \text{ about a fixed point } O,$$
  

$$\mathbf{M}^{C} = \dot{\mathbf{H}}^{C} \text{ about the center of mass } C,$$
  

$$\mathbf{M}^{P} = \dot{\mathbf{H}}^{P} + (\mathbf{v}_{P} - \mathbf{v}_{C}) \times \mathbf{G} = \dot{\mathbf{H}}^{C} + (\mathbf{r}_{C} - \mathbf{r}_{P}) \times m\mathbf{a}_{C} \text{ about a material point } P \text{ on } \mathcal{B}.$$
(11)

where the angular momentum of a rigid body  $\mathcal{B}$  relative to any material point P on the body is

$$\mathbf{H}^{P} = \int_{\mathcal{B}} (\mathbf{r} - \mathbf{r}_{P}) \times \mathbf{v} dm = \mathbf{H}^{C} + (\mathbf{r}_{C} - \mathbf{r}_{P}) \times \mathbf{G}, \quad \text{where} \quad \mathbf{H}^{C} = \int_{B} (\mathbf{r} - \mathbf{r}_{C}) \times \mathbf{v} dm.$$
(12)

Letting  $\mathbf{r} - \mathbf{r}_C = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_Z$  can calculate

$$\begin{bmatrix} \mathbf{H}^{C} \cdot \mathbf{e}_{x} \\ \mathbf{H}^{C} \cdot \mathbf{e}_{y} \\ \mathbf{H}^{C} \cdot \mathbf{e}_{z} \end{bmatrix} = \begin{bmatrix} I_{xx}^{C} & I_{xy}^{C} & I_{xz}^{C} \\ I_{xy}^{C} & I_{yy}^{C} & I_{yz}^{C} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \cdot \mathbf{e}_{x} \\ \boldsymbol{\omega} \cdot \mathbf{e}_{y} \\ \boldsymbol{\omega} \cdot \mathbf{e}_{z} \end{bmatrix}$$
(13)

where

$$I_{xx}^{C} = \int_{B} (y^{2} + z^{2}) dm, \quad I_{yy}^{C} = \int_{B} (x^{2} + z^{2}) dm, \quad I_{zz}^{C} = \int_{B} (x^{2} + y^{2}) dm, \quad I_{xy}^{C} = -\int_{B} xy dm, \quad I_{yz}^{C} = -\int_{B} yz dm, \quad I_{xz}^{C} = -\int_{B} xz dm$$
(14)

Thus, for  $\boldsymbol{\omega} = \dot{\theta} \mathbf{E}_z$ , we have

$$\dot{\mathbf{H}}^{O} = (I_{xz}^{O}\dot{\omega} - I_{yz}^{O}\omega^{2})\mathbf{e}_{x} + (I_{yz}^{O}\dot{\omega} + I_{xz}^{O}\omega^{2})\mathbf{e}_{y} + I_{zz}^{O}\dot{\omega}\mathbf{E}_{z}, 
\dot{\mathbf{H}}^{C} = (I_{xz}^{C}\dot{\omega} - I_{yz}^{C}\omega^{2})\mathbf{e}_{x} + (I_{yz}^{C}\dot{\omega} + I_{xz}^{C}\omega^{2})\mathbf{e}_{y} + I_{zz}^{C}\dot{\omega}\mathbf{E}_{z}, 
\dot{\mathbf{H}}^{P} = (I_{xz}^{P}\dot{\omega} - I_{yz}^{P}\omega^{2})\mathbf{e}_{x} + (I_{yz}^{P}\dot{\omega} + I_{xz}^{P}\omega^{2})\mathbf{e}_{y} + I_{zz}^{P}\dot{\omega}\mathbf{E}_{z} + \frac{d}{dt}\left((\mathbf{r}_{C} - \mathbf{r}_{P}) \times m\mathbf{v}_{P}\right).$$
(15)

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The angular impulse - angular momentum equations are equivalently

$$\int_{t_A}^{t_B} \mathbf{M}^C dt = \mathbf{H}^C(t_B) - \mathbf{H}^C(t_A) \quad \text{where } C \text{ is the center of mass,}$$

$$\int_{t_A}^{t_B} \mathbf{M}^O dt = \mathbf{H}^O(t_B) - \mathbf{H}^O(t_A) \quad \text{if there is a fixed point } O.$$
(16)

**Parallel axis theorem** Consider a material point A on the rigid body such that  $\mathbf{r}_A - \mathbf{r}_C = A_x \mathbf{e}_x + A_y \mathbf{e}_y + A_z \mathbf{e}_z$ , then according to the parallel axis theorem

$$I_{xx}^{A} = I_{xx}^{C} + m(A_{y}^{2} + A_{z}^{2}), \quad I_{xy}^{A} = I_{xy}^{C} - mA_{x}A_{y}, \quad etc.$$
(17)

The Koenig decomposition for the kinetic energy of a RB is

$$T = \frac{1}{2}m\mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2}\mathbf{H}^C \cdot \boldsymbol{\omega}.$$
(18)

If the body has a point O with zero velocity,  $\mathbf{v}_O = \mathbf{0}$ , then

$$T = \frac{1}{2}m\mathbf{H}^O \cdot \boldsymbol{\omega}.$$
 (19)

The work-energy theorem for a RB between  $[t_A, t_B]$  is

$$T_B - T_A = \sum_{i=1}^{K} W_{\mathbf{F}_i,AB} + W_{\mathbf{M}_e,AB},$$
(20)

where the work of a force  $\mathbf{F}_i$  applied at a point with position vector  $\mathbf{r}_i$  over the interval  $[t_A, t_B]$  is

$$W_{\mathbf{F}_{i},AB} = \int_{t_{A}}^{t_{B}} \mathbf{F}_{i} \cdot \mathbf{v}_{i} dt = \int_{\mathbf{r}_{i}(t_{A})}^{\mathbf{r}_{i}(t_{B})} \mathbf{F} \cdot d\mathbf{r}_{i}$$
(21)

and the work of a moment is

$$W_{\mathbf{M},AB} = \int_{t_A}^{t_B} \mathbf{M} \cdot \boldsymbol{\omega} dt.$$
(22)

Conservative Force A conservative  $\mathbf{F}_c$  is such that  $W_{\mathbf{F}_c,AB} = -(U_B - U_A)$ . Examples include

- any constant force  $\mathbf{C}$  with  $U = -\mathbf{C} \cdot \mathbf{r}$ ,
- the gravitational force  $\mathbf{F}_G = G \frac{M_e m}{(R_e+h)^2} (-\mathbf{e}_r)$  with  $U = -\frac{GM_e m}{r}$ , and
- the spring force  $\mathbf{F}_s = -K\varepsilon \frac{\mathbf{r}-\mathbf{r}_A}{||\mathbf{r}-\mathbf{r}_A||}$  with  $U = \frac{1}{2}K\varepsilon^2$ . The spring stretch  $\varepsilon = ||\mathbf{r}-\mathbf{r}_A|| \ell_0$ .