Kinetics of Rigid Bodies

Theresa Honein

November 2024

1 The Balance Laws for a rigid body

The balance laws for a rigid body are Euler's first law, also known as the Balance of Linear Momentum,

$$\mathbf{F} = m\mathbf{a}, \tag{1}$$

and Euler's second law, also known as the Balance of Angular Momentum which has three equivalent forms:

$$\mathbf{M}^O = \dot{\mathbf{H}}^O \quad \text{about a fixed point } O, \tag{2}$$

$$\mathbf{M}^C = \dot{\mathbf{H}}^C \quad \text{about the center of mass } C, \tag{3}$$

$$\mathbf{M}^P = \dot{\mathbf{H}}^P + (\mathbf{v}_P - \mathbf{v}_C) \times \mathbf{G} = \dot{\mathbf{H}}^C + (\mathbf{r}_C - \mathbf{r}_P) \times m\mathbf{a}_C \quad \text{about any material point } P \text{ on the body.} \tag{4}$$

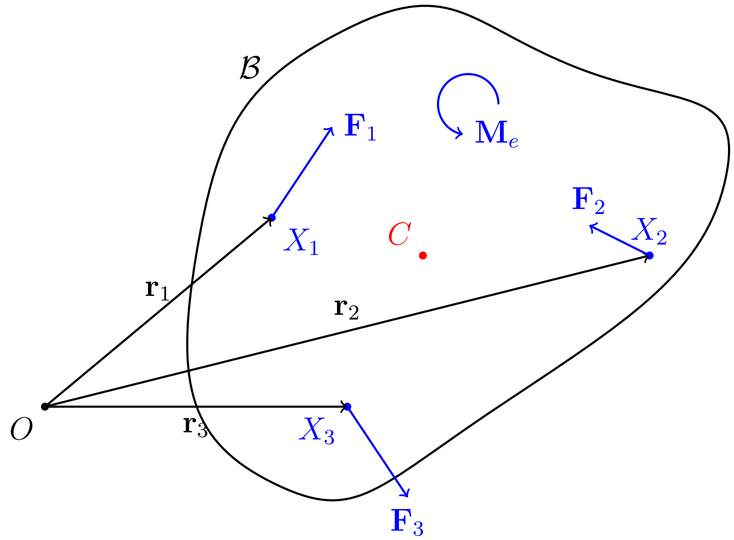
where O is a fixed point. Here, the BoAM is an independent postulate, not derivable from the BoLM.

1.1 Resultant forces and resultant moments

The resultant force $\mathbf{F}$ acting on the rigid body is the sum of all forces acting on the rigid body.

The resultant moment relative to a fixed point O, $\mathbf{M}^{\mathbf{O}}$ is the resultant external moment relative to O of all of the moments acting on the rigid body. These moments may be decomposed into two additive parts: the moment due to the individual external forces acting on the rigid body and applied external moments that are not due to external forces.

Consider a system of forces and moments acting on rigid body. A set of K forces $\mathbf{F}_i$, $(i = 1, \ldots, K)$ act on the rigid body. The force $\mathbf{F}_i$ acts at the material points X_i, that has a position vector $\mathbf{x}_i$. In addition, a moment $\mathbf{M}_e$, that is not due to the moment of an applied forces, acts on the rigid body.



For this system of applied forces and moments, the resultants are

$$\begin{aligned}\mathbf{F} &= \sum_{i=1}^{K} \mathbf{F}_i, \\ \mathbf{M}^O &= \mathbf{M}_e + \sum_{i=1}^{K} \mathbf{r}_i \times \mathbf{F}_i, \\ \mathbf{M}^C &= \mathbf{M}_e + \sum_{i=1}^{K} (\mathbf{r}_i - \mathbf{r}_c) \times \mathbf{F}_i.\end{aligned} \tag{5}$$

Examples of pure moments:

1. reaction moments $\mathbf{M}_R$ at joints.
2. moments $-K_T(\theta - \theta_0)\mathbf{E}_z$ supplied by torsional springs.

1.1.1 Does the weight give a moment about the center of mass?

We calculate the total weight force of the body by continuously summing (integrating) the weights of the differential masses dm

$$\mathbf{W} = \int_{\mathcal{B}} \mathbf{g} dm = \mathbf{g} \int_{\mathcal{B}} dm = m\mathbf{g}, \tag{6}$$

ad we sum the moments of these differential forces about the center of mass

$$\mathbf{M}^C = \int_{\mathcal{B}} \boldsymbol{\pi} \times \mathbf{g} dm = \left(\int_{\mathcal{B}} \boldsymbol{\pi} dm \right) \times \mathbf{g} = \mathbf{0} \tag{7}$$

since $\int_{\mathcal{B}} \pi dm = \mathbf{0}$.

1.2 Equivalence of the BoAM forms

The following developments are true for rigid bodies where the inertia matrix components are constant.

1.3 BoAM about any point P

Starting with the BoAM about the center of mass

$$\mathbf{M}^C = \dot{\mathbf{H}}^C, \tag{8}$$

we replace $\mathbf{M}^P = \mathbf{M}^C + (\mathbf{r}_C - \mathbf{r}_P) \times \mathbf{F}$ to get

$$\mathbf{M}^P = \dot{\mathbf{H}}^C + (\mathbf{r}_C - \mathbf{r}_P) \times \mathbf{F}. \tag{9}$$

thus recovering equation 4_2. To recover equation 4_1, recall $\mathbf{H}^C = \mathbf{H}^P + (\mathbf{r}_P - \mathbf{r}_C) \times \mathbf{G}$ and calculate

$$\dot{\mathbf{H}}^C = \dot{\mathbf{H}}^P + (\mathbf{v}_P - \mathbf{v}_C) \times \mathbf{G} + (\mathbf{r}_P - \mathbf{r}_C) \times m\mathbf{a}_C, \tag{10}$$

and replace it in equation 9 to obtain

$$\begin{aligned} \mathbf{M}^P &= \dot{\mathbf{H}}^P + (\mathbf{v}_P - \mathbf{v}_C) \times \mathbf{G} + (\mathbf{r}_P - \mathbf{r}_C) \times m\mathbf{a}_C + (\mathbf{r}_C - \mathbf{r}_P) \times \mathbf{F}, \\ &= \dot{\mathbf{H}}^P + (\mathbf{v}_P - \mathbf{v}_C) \times \mathbf{G}. \end{aligned} \tag{11}$$

1.3.1 BoAM about the a fixed point O

If P is a fixed point O, then $\mathbf{v}_O = \mathbf{0}$ and equation 11 simplifies to

$$\mathbf{M}^O = \dot{\mathbf{H}}^O$$

thus recovering equation $\mathbf{M}^O$.

1.4 Calculating $\dot{\mathbf{H}}$

Recall

$$\mathbf{H}^C = (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\mathbf{e}_x + (I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\mathbf{e}_y + (I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z)\mathbf{e}_z. \tag{12}$$

where

$$\boldsymbol{\omega} = \omega_x\mathbf{e}_x + \omega_y\mathbf{e}_y + \omega_z\mathbf{e}_z. \tag{13}$$

We need to calculate $\dot{\mathbf{H}}^C$.

$$\dot{\mathbf{H}}^C = \overset{\circ}{\mathbf{H}}{}^C + \boldsymbol{\omega} \times \mathbf{H}^C. \tag{14}$$

where $\overset{\circ}{\mathbf{H}}{}^C$ is the corotational rate of $\mathbf{H}$, that is the time derivative of $\mathbf{H}$ that is obtained while keeping $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ fixed:

$$\mathbf{H}^C = (I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z)\mathbf{e}_x + (I_{xy}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z)\mathbf{e}_y + (I_{xz}\dot{\omega}_x + I_{yz}\dot{\omega}_y + I_{zz}\dot{\omega}_z)\mathbf{e}_z. \tag{15}$$

Useful identity:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}. \tag{16}$$

1.5 The fixed axis rotation case

In this case, it is usually more convenient to calculate the BoAM about a fixed point O. Here, we can define

$$\begin{aligned}\mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y,\\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{e}_x,\\ \mathbf{e}_z &= \mathbf{E}_z,\end{aligned} \tag{17}$$

$$\begin{aligned}\dot{\mathbf{e}}_x &= \dot{\theta}\mathbf{e}_y,\\ \dot{\mathbf{e}}_y &= -\dot{\theta}\mathbf{e}_x,\end{aligned} \tag{18}$$

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z = \omega\mathbf{E}_z. \tag{19}$$

$$\mathbf{H}^O = I_{xz}^O\omega\mathbf{e}_x + I_{yz}^O\omega\mathbf{e}_y + I_{zz}^O\omega\mathbf{E}_z, \tag{20}$$

$$\dot{\mathbf{H}}^O = (I_{xz}^O\dot{\omega} - I_{yz}^O\omega^2)\mathbf{e}_x + (I_{yz}^O\dot{\omega} + I_{xz}^O\omega^2)\mathbf{e}_y + I_{zz}^O\dot{\omega}\mathbf{E}_z. \tag{21}$$

The balance laws for the fixed axis of rotation case can be written as

$$\mathbf{F} = m\dot{\mathbf{v}}_c, \tag{22}$$

$$\mathbf{M}^O = (I_{xz}^O\dot{\omega} - I_{yz}^O\omega^2)\mathbf{e}_x + (I_{yz}^O\dot{\omega} + I_{xz}^O\omega^2)\mathbf{e}_y + I_{zz}^O\dot{\omega}\mathbf{E}_z. \tag{23}$$

1.6 General Plane Motion

For general plane motion, we still have

$$\begin{aligned}\mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y,\\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{e}_x,\\ \mathbf{e}_z &= \mathbf{E}_z,\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{\mathbf{e}}_x &= \dot{\theta}\mathbf{e}_y,\\ \dot{\mathbf{e}}_y &= -\dot{\theta}\mathbf{e}_x,\end{aligned} \tag{25}$$

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z = \omega\mathbf{E}_z. \tag{26}$$

Calculating the balance of angular momentum about the center of mass, we have

$$\mathbf{H}^C = I_{xz}^C\omega\mathbf{e}_x + I_{yz}^C\omega\mathbf{e}_y + I_{zz}^C\omega\mathbf{E}_z, \tag{27}$$

$$\dot{\mathbf{H}}^C = (I_{xz}^C\dot{\omega} - I_{yz}^C\omega^2)\mathbf{e}_x + (I_{yz}^C\dot{\omega} + I_{xz}^C\omega^2)\mathbf{e}_y + I_{zz}^C\dot{\omega}\mathbf{E}_z. \tag{28}$$

Then, the balance laws become

$$\mathbf{F} = m\dot{\mathbf{v}}_c, \tag{29}$$

$$\mathbf{M}^C = (I_{xz}^C\dot{\omega} - I_{yz}^C\omega^2)\mathbf{e}_x + (I_{yz}^C\dot{\omega} + I_{xz}^C\omega^2)\mathbf{e}_y + I_{zz}^C\dot{\omega}\mathbf{E}_z. \tag{30}$$

About any point P, we have

$$\mathbf{H}^P = I^P_{xz}\omega\mathbf{e}_x + I^P_{yz}\omega\mathbf{e}_y + I^P_{zz}\omega\mathbf{E}_z, \tag{31}$$
$$\dot{\mathbf{H}}^P = (I^P_{xz}\dot{\omega} - I^P_{yz}\omega^2)\mathbf{e}_x + (I^P_{yz}\dot{\omega} + I^P_{xz}\omega^2)\mathbf{e}_y + I^P_{zz}\dot{\omega}\mathbf{E}_z. \tag{32}$$

then, the balance laws become

$$\mathbf{F} = m\dot{\mathbf{v}}_c, \tag{33}$$
$$\mathbf{M}^C = (I^P_{xz}\dot{\omega} - I^P_{yz}\omega^2)\mathbf{e}_x + (I^P_{yz}\dot{\omega} + I^P_{xz}\omega^2)\mathbf{e}_y + I^P_{zz}\dot{\omega}\mathbf{E}_z + (\mathbf{v}_P - \mathbf{v}_C) \times \mathbf{G}, \tag{34}$$
$$= (I^C_{xz}\dot{\omega} - I^C_{yz}\omega^2)\mathbf{e}_x + (I^C_{yz}\dot{\omega} + I^C_{xz}\omega^2)\mathbf{e}_y + I^C_{zz}\dot{\omega}\mathbf{E}_z + (\mathbf{r}_C - \mathbf{r}_P) \times m\mathbf{a}_C. \tag{35}$$

2 Work-Energy Theorem and Energy Conservation

Here, we first show the Koenig decomposition for the kinetic energy of a rigid body:

$$T = \frac{1}{2}m\mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2}\mathbf{H} \cdot \boldsymbol{\omega}. \tag{36}$$

This is then followed by a development of the work-energy theorem for a rigid body:

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v}_C + \mathbf{M} \cdot \boldsymbol{\omega} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_e \cdot \boldsymbol{\omega}. \tag{37}$$

As in particles and systems of particles, this theorem can be used to establish conservation of the total energy of a rigid body during a motion.

2.1 Koenig's Decomposition

By definition, the kinetic energy T of a rigid body is

$$T = \frac{1}{2}\int_R \mathbf{v} \cdot \mathbf{v}\rho dv. \tag{38}$$

where

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_C + \boldsymbol{\omega} \times \boldsymbol{\pi}, \\ \boldsymbol{\pi} &= \mathbf{r} - \mathbf{r}_C, \\ \boldsymbol{\omega} &= \omega_x\mathbf{e}_x + \omega_y\mathbf{e}_y + \omega_z\mathbf{e}_z. \end{aligned} \tag{39}$$

Substituting

$$T = \frac{1}{2}\int_{\mathcal{B}} (\mathbf{v}_C \cdot \mathbf{v}_C + 2\mathbf{v}_C \cdot (\boldsymbol{\omega} \times \boldsymbol{\pi}) + (\boldsymbol{\omega} \times \boldsymbol{\pi}) \cdot (\boldsymbol{\omega} \times \boldsymbol{\pi}))\, dm \tag{40}$$

However,

$$\frac{1}{2}\int_{\mathcal{B}} \mathbf{v}_C \cdot \mathbf{v}_C dm = \frac{\mathbf{v}_C \cdot \mathbf{v}_C}{2}\int dm = \frac{1}{2}m\mathbf{v}_C \cdot \mathbf{v}_C. \tag{41}$$
$$\int_{\mathcal{B}} \mathbf{v}_C \cdot (\boldsymbol{\omega} \times \boldsymbol{\pi})dm = \mathbf{v}_C \cdot \left(\boldsymbol{\omega} \times \int \pi dm\right) = 0. \tag{42}$$

Thus,

$$T = \frac{1}{2} m \mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2} \int_{\mathcal{B}} (\boldsymbol{\omega} \times \boldsymbol{\pi}) \cdot (\boldsymbol{\omega} \times \boldsymbol{\pi}) dm \tag{43}$$

we can simplify

$$(\boldsymbol{\omega} \times \boldsymbol{\pi}) \cdot (\boldsymbol{\omega} \times \boldsymbol{\pi}) = ((\boldsymbol{\pi} \cdot \boldsymbol{\pi}) \boldsymbol{\omega} - (\boldsymbol{\pi} \cdot \boldsymbol{\omega}) \cdot \boldsymbol{\omega}). \tag{44}$$

Recall that

$$\mathbf{H}^C = \int_{\mathcal{B}} \boldsymbol{\pi} \times (\omega \times \boldsymbol{\pi}) dm = \int_{\mathcal{B}} ((\boldsymbol{\pi} \cdot \boldsymbol{\pi}) \boldsymbol{\omega} - (\boldsymbol{\pi} \cdot \boldsymbol{\omega}) \boldsymbol{\pi}) dm. \tag{45}$$

Hence, we obtain the Koenig decomposition:

$$T = \frac{1}{2} m \mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2} \mathbf{H} \cdot \boldsymbol{\omega}. \tag{46}$$

2.2 The Work-Energy Theorem

$$\dot{T} = \frac{1}{2} m \dot{\mathbf{v}}_C \cdot \mathbf{v}_C + \frac{1}{2} m \mathbf{v}_C \cdot \dot{\mathbf{v}}_C + \frac{1}{2} \dot{\mathbf{H}} \cdot \boldsymbol{\omega} + \frac{1}{2} \mathbf{H} \cdot \dot{\boldsymbol{\omega}}. \tag{47}$$

We need to show that $\dot{\mathbf{H}} \cdot \boldsymbol{\omega} = \mathbf{H} \cdot \dot{\boldsymbol{\omega}}$.

$$\begin{aligned}
\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} &= \frac{d}{dt} (\omega_x \mathbf{e}_x + \omega_y \mathbf{e}_y + \omega_z \mathbf{e}_z) && (48) \\
&= \dot{\omega}_x \mathbf{e}_x + \dot{\omega}_y \mathbf{e}_y + \dot{\omega}_z \mathbf{e}_z + \omega_x \dot{\mathbf{e}}_x + \omega_y \dot{\mathbf{e}}_y + \omega_z \dot{\mathbf{e}}_z && (49) \\
&= \dot{\omega}_x \mathbf{e}_x + \dot{\omega}_y \mathbf{e}_y + \dot{\omega}_z \mathbf{e}_z + \boldsymbol{\omega} \times (\omega_x \mathbf{e}_x + \omega_y \mathbf{e}_y + \omega_z \mathbf{e}_z) && (50) \\
&= \dot{\omega}_x \mathbf{e}_x + \dot{\omega}_y \mathbf{e}_y + \dot{\omega}_z \mathbf{e}_z. && (51)
\end{aligned}$$

Another direct calculation using this expression for $\boldsymbol{\alpha}$ shows that

$$\mathbf{H}^C \cdot \dot{\boldsymbol{\omega}} = (I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z) \dot{\omega}_x + (I_{xy} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z) \dot{\omega}_y + (I_{xz} \omega_x + I_{yz} \omega_y + I_{zz} \omega_z) \dot{\omega}_z. \tag{52}$$

Comparing this to the corresponding expression for $\dot{H}^C \cdot \boldsymbol{\omega}$, we find that they are equal. Consequently,

$$\dot{T} = \frac{1}{2} m \dot{\mathbf{v}}_C \cdot \mathbf{v}_C + \frac{1}{2} m \mathbf{v}_C \cdot \dot{\mathbf{v}}_C + \frac{1}{2} \dot{\mathbf{H}} \cdot \boldsymbol{\omega} + \frac{1}{2} \dot{\mathbf{H}} \cdot \boldsymbol{\omega}. \tag{53}$$

This result implies that

$$\dot{T} = m \mathbf{v}_C \cdot \mathbf{v}_C + \dot{\mathbf{H}} \cdot \boldsymbol{\omega}. \tag{54}$$

Invoking the balance of linear momentum and the balance of angular momentum, we obtain the work-energy theorem:

$$\dot{T} = \mathbf{F} \cdot \mathbf{v}_C + \mathbf{M} \cdot \boldsymbol{\omega}. \tag{55}$$

You should notice how this is a natural extension of the work-energy theorem for a single particle.

2.3 An alternative form of the work-energy theorem

Recall,

$$\mathbf{F} = \sum_{i=1}^{K} \mathbf{F}_i \tag{56}$$

$$\mathbf{M}^C = \mathbf{M}_e + \sum_{i=1}^{K} (\mathbf{r}_i - \mathbf{r}_C) \times \mathbf{F}_i \tag{57}$$

[continue the 9.2.3.]

3 Integral Forms of the Balance Laws

4 Summary

[Copied From OOR Primer Chapter 9]

For a system of K forces $\mathbf{F}_i$, $(i = 1, \ldots, K)$ and a moment $\mathbf{M}_e$, that is not due to the moment of an applied force, acting on the rigid body, the resultant force $\mathbf{F}$ and moments are

$$\begin{aligned}
\mathbf{F} &= \sum_{i=1}^{K} \mathbf{F}_i, \\
\mathbf{M}^O &= \mathbf{M}_e + \sum_{i=1}^{K} \mathbf{r}_i \times \mathbf{F}_i, \\
\mathbf{M}^C &= \mathbf{M}_e + \sum_{i=1}^{K} (\mathbf{r}_i - \mathbf{r}_C) \times \mathbf{F}_i,
\end{aligned} \tag{58}$$

where $\mathbf{M}^O$ is the resultant moment relative to a fixed point O and $\mathbf{M}$ is the resultant moment relative to the center of mass C of the rigid body.

The relationship between forces and moments and the motion of the rigid body is postulated using the balance laws. There are two equivalent sets of balance laws:

$$\mathbf{F} = m\frac{d\mathbf{v}_C}{dt}, \quad \text{and} \quad \mathbf{M}^O = \dot{\mathbf{H}}^O. \tag{59}$$

and

$$\mathbf{F} = m\frac{d\mathbf{v}_C}{dt}, \quad \text{and} \quad \mathbf{M}^C = \dot{\mathbf{H}}^C. \tag{60}$$

When these balance laws are specialized to the case of a fixed-axis rotation, the expressions for $\dot{\mathbf{H}}^O$ and $\dot{\mathbf{H}}^C$ simplify. For instance,

$$\begin{aligned}
&\mathbf{F} = m\dot{\mathbf{v}}_C, \\
&\mathbf{M} = (I_{xz}\dot{\omega} - I_{yz}\omega^2)\mathbf{e}_x + (I_{yz}\dot{\omega} + I_{xz}\omega^2)\mathbf{e}_y + I_{zz}\dot{\omega}\mathbf{e}_z.
\end{aligned} \tag{61}$$

In most problems, $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ are chosen such that $I_{xz} = I_{yz} = 0$.

To establish conservations of energy, two equivalent forms of the work-energy theorem were developed. First, however, the Koenig decomposition for the kinetic energy of a rigid body was established:

$$T = \frac{1}{2}m\mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2}\mathbf{H} \cdot \boldsymbol{\omega}.$$

This was then followed by a development of the work-energy theorem for a rigid body:

$$\frac{dT}{dt} = \mathbf{F} \cdot [\mathbf{v}]_{\mathbf{C}} + \mathbf{M} \cdot \boldsymbol{\omega} = \sum_{\mathbf{i=1}}^{\mathbf{K}} \mathbf{F_i} \cdot \mathbf{v_i} + \mathbf{M_e} \cdot \boldsymbol{\omega}. \tag{62}$$

To establish energy conservation results, this theorem is used in a similar manner to the one employed with particles and systems of particles.

Four sets of applications were the discussed:

1. Purely translational motion of a rigid body where $\boldsymbol{\omega} = \boldsymbol{\alpha} = \mathbf{0}$.
2. A rigid body with a fixed point O.
3. Rolling rigid bodies and sliding rigid bodies.
4. Imbalanced rotors.

It is important to note that for the second set of applications, the balance law $\mathbf{M}^O = \dot{\mathbf{H}}^O$ is more convenient to use than $\mathbf{M} = \dot{\mathbf{H}}$. The role of $\mathbf{M}_R$ in these problems is to ensure that the axis of rotation remains $\mathbf{E}_z$. Finally, the four steps discussed are used as a guide to solving all of the applications.