Mass Center, Linear Momentum, Angular Momentum

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1 The Center of Mass

The center of mass C of a body $\mathcal{B}$ has position vector

$$\mathbf{r}_C = \frac{\int_{\mathcal{B}} \mathbf{r} dm}{\int_{\mathcal{B}} dm}, \tag{1}$$

with $dm = \rho(x,t)dv$ where ρ is the mass density for unit volume and dv.

A special feature for rigid bodies is that the center of mass behaves as if it were a material point.

$$\begin{aligned}
&\mathbf{v}_C - \mathbf{v}_A = \boldsymbol{\omega} \times (\mathbf{r}_C - \mathbf{r}_A), \\
&\mathbf{a}_C - \mathbf{a}_A = \boldsymbol{\alpha} \times (\mathbf{r}_C - \mathbf{r}_A) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_C - \mathbf{r}_A)).
\end{aligned}$$

Example: Find the center of mass of any body from Set 15.

2 Linear Momentum

By definition, the linear momentum of a body $\mathbf{G}$ is the sum of the linear momenta of its constituents

$$\mathbf{G} = \int_B \mathbf{v} dm.$$

Equivalently,

$$\mathbf{G} = \int_{\mathcal{B}} \mathbf{v} dm = \int_{\mathcal{B}} \frac{d\mathbf{r}}{dt} dm = \frac{d}{dt}\left(\int_{\mathcal{B}} \mathbf{r} dm\right) = \frac{d}{dt}(m\mathbf{r}_C) = m\mathbf{v}_C.$$

3 Kinematics of Rolling and Sliding

Consider a rigid body $\mathcal{B}$ that is in contact with a fixed surface $\mathcal{S}$. As the body moves on the fixed surface, the material point of the body that is is contact with the surface may change. We denote the material point of the body that is in contact at time t by P with position vector $\mathbf{r}_P$ and velocity $\mathbf{v}_P$ and the unit normal to $\mathcal{S}$ by $\mathbf{n}$.

Since P is a material point of $\mathcal{B}$, we can write

$$\mathbf{v}_P - \mathbf{v}_C = \boldsymbol{\omega} \times (\mathbf{r}_P - \mathbf{r}_C),$$
$$\mathbf{a}_P - \mathbf{a}_C = \boldsymbol{\alpha} \times (\mathbf{r}_P - \mathbf{r}_C) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_P - \mathbf{r}_C)).$$

For a rigid body that is sliding on the fixed surface $\mathcal{S}$, the component of $\mathbf{v}_P$ in the $\mathbf{n}$ direction is zero:

$$\mathbf{v}_P \cdot \mathbf{n} = \mathbf{0}.$$

Thus,

$$\mathbf{v}_C = -\boldsymbol{\omega} \times (\mathbf{r}_P - \mathbf{r}_C) \cdot \mathbf{n}.$$

For a rigid body that is rolling on the fixed surface $\mathcal{S}$, the velocity of the instantaneous point of contact P is zero:

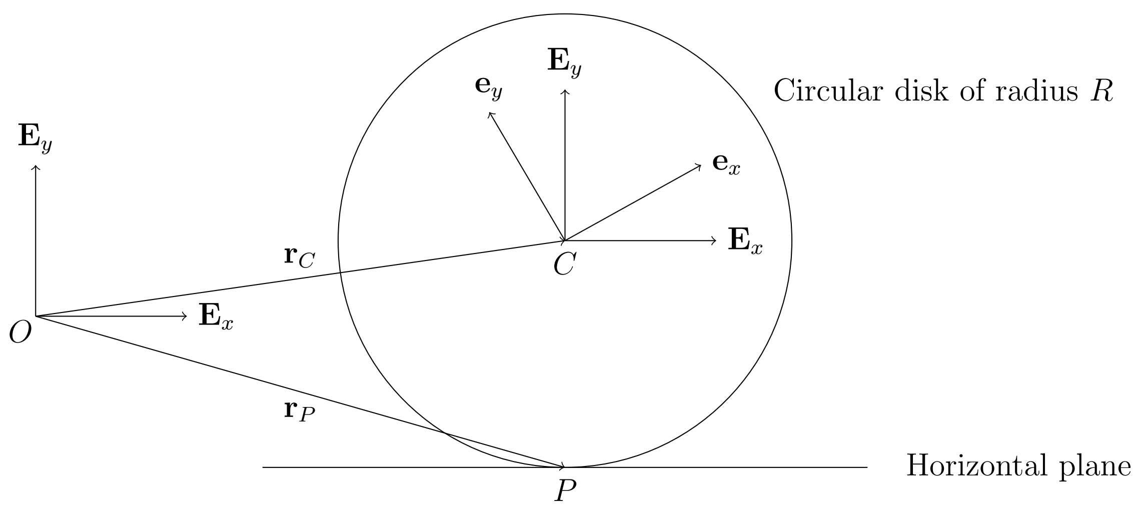
$$\mathbf{v}_P = \mathbf{0}.$$

Hence,

$$\mathbf{v}_C = -\boldsymbol{\omega} \times (\mathbf{r}_P - \mathbf{r}_C).$$

Note that the acceleration $\mathbf{a}_P$ is not necessarily $\mathbf{0}$.

4 Kinematics of a Rolling Circular Disk



Consider an upright homogeneous disk of radius R that is rolling on a plane. We define a corotational basis for the disk

$$\mathbf{e}_x = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y,$$
$$\mathbf{e}_y = \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_y,$$
$$\mathbf{e}_z = \mathbf{E}_z.$$

Because the motion is a fixed axis rotation,

5 Angular Momenta

The angular momentum of a system relative to any point P is

$$\mathbf{H}^P = \int_B (\mathbf{r} - \mathbf{r}_P) \times \mathbf{v} dm.$$

For P being the center of mass C and the fixed origin O, receptively we have

$$\mathbf{H}^C = \int_B (\mathbf{r} - \mathbf{r}_C) \times \mathbf{v} dm,$$
$$\mathbf{H}^O = \int_B \mathbf{r} \times \mathbf{v} dm, .$$

$\mathbf{H}^P$ and $\mathbf{H}^C$ are related through

$$\mathbf{H}^P = \mathbf{H}^C + (\mathbf{r}_C - \mathbf{r}_P) \times \mathbf{G}.$$

for the special case that P is the origin O, we get

$$\mathbf{H}^O = \mathbf{H}^C + \mathbf{r}_C \times \mathbf{G}.$$

In other words, the angular momentum of a rigid body relative to a fixed point O is the sum of the angular momentum of the rigid body about its center of mass and the angular momentum of its center of mass relative to O.

5.1 Inertia Tensors

Recall, for any material points on a body, its velocity $\mathbf{v}$ is

$$\mathbf{v} - \mathbf{v}_C = \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_C).$$

Also, define

$$\boldsymbol{\pi} = \mathbf{r} - \mathbf{r}_C = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z.$$

Then,

$$\begin{aligned}
\mathbf{H}^C &= \int_B (\mathbf{r} - \mathbf{r}_C) \times \mathbf{v} dm, \\
&= \int_B (\mathbf{r} - \mathbf{r}_C) \times (\mathbf{v}_C + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_C)) dm \\
&= \int_B \boldsymbol{\pi} \times (\mathbf{v}_C + \boldsymbol{\omega} \times \boldsymbol{\pi}) dm, \\
&= \int_B \boldsymbol{\pi} \times \mathbf{v} dm + \int_B \boldsymbol{\pi} \times (\boldsymbol{\omega} \times \boldsymbol{\pi}) dm
\end{aligned}$$

The first integral is zero, and from the BAC-CAB identity, the second integral can be written as

$$\mathbf{H}^C = \int_B ((\boldsymbol{\pi} \cdot \boldsymbol{\pi})\boldsymbol{\omega} - (\boldsymbol{\pi} \cdot \boldsymbol{\omega})\boldsymbol{\pi}) dm$$

Substituting

$$\mathbf{r} - \mathbf{r}_C = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z,$$
$$\boldsymbol{\omega} = \omega_x\mathbf{e}_x + \omega_y\mathbf{e}_y + \omega_z\mathbf{e}_z,$$

we get

$$\mathbf{H} = (I^C_{xx}\omega_x + I^C_{xy}\omega_y + I^C_{xz}\omega_z)\mathbf{e}_x + (I^C_{xy}\omega_x + I^C_{yy}\omega_y + I^C_{yz}\omega_z)\mathbf{e}_y + (I^C_{xz}\omega_x + I^C_{yz}\omega_y + I^C_{zz}\omega_z)\mathbf{e}_z.$$

where

$$I^C_{xx} = \int_B (y^2 + z^2)dm,$$
$$I^C_{xy} = -\int_B xydm$$

In matrix form,

$$\begin{bmatrix} \mathbf{H}^C \cdot \mathbf{e}_x \\ \mathbf{H}^C \cdot \mathbf{e}_y \\ \mathbf{H}^C \cdot \mathbf{e}_z \end{bmatrix} = \begin{bmatrix} I^C_{xx} & I^C_{xy} & I^C_{xz} \\ I^C_{xy} & I^C_{yy} & I^C_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \cdot \mathbf{e}_x \\ \boldsymbol{\omega} \cdot \mathbf{e}_y \\ \boldsymbol{\omega} \cdot \mathbf{e}_z \end{bmatrix}$$

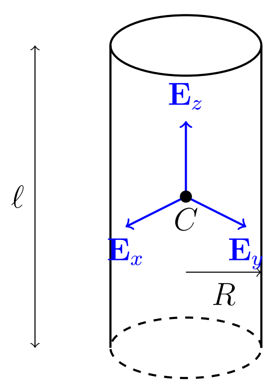
The matrix in this equation is known as the inertia matrix. Its diagonal components are terms the moments of inertia and its nondiagonal components are termed the products of inertia.

When the corotational basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is an eigenbasis of the inertia matrix, then the products of inertia are zero.

Example: Watch video: Professor Walter Lewin example with rolling cylinders

Example: Tight rope walker A tight rope walker holds a pole to increase his or her moment of inertia against rotating off of rope. This gives the tight rope walker more time to react.

5.1.1 Example: Solid Cylinder



Noting that $dm = \rho dV = \rho r dr d\theta dz$, we have

$$\begin{aligned}
I_{zz}^C &= \int_{\mathcal{B}} (x^2 + y^2) dm \\
&= \int_{\mathcal{B}} r^2 dm \\
&= \rho \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \int_0^{2\pi} \int_0^R r^3 dr d\theta dz \\
&= \rho 2\pi\ell \int_0^R r^3 dr \\
&= \frac{1}{2}\rho\pi\ell R^4
\end{aligned} \tag{2}$$

Since $V = \pi R^2 \ell$ and $m = \rho\pi\ell R^4$, we can simplify this expression to $I_{xx}^C = \frac{mR^2}{2}$.

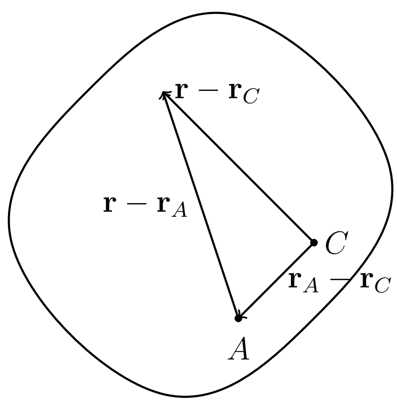
5.1.2 Example: Cylindrical Hoop

$$I_{zz}^C = \int_{\mathcal{B}} r^2 dm = R^2 \int_{\mathcal{B}} dm = mR^2 \tag{3}$$

since every dm has $r = R$.

5.2 The parallel axis theorem

Lets say we already have the $I_{xx}^C, I_{xy}^C, I_{xz}^C, etc$ about the center of mass C. Now I want $I_{xx}^A, I_{xy}^A, I^A xz, etc$ about a fixed point A on the rigid body where the axes are parallel.



Recall that

$$\begin{aligned}
I_{xx} &= \int_{\mathcal{B}} (x^2 + y^2) dm, \\
I_{xy} &= \int_{\mathcal{B}} (xy) dm, \\
\mathbf{r} - \mathbf{r}_C &= x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z,
\end{aligned}$$

and let

$$\mathbf{r}_A - \mathbf{r}_C = A_x\mathbf{e}_x + A_z\mathbf{e}_y + A_z\mathbf{e}_z.$$

Then,

$$\begin{aligned} I_{xx}^A &= \int_{\mathcal{B}} [(y - y_A)^2 + (z - A_z)^2] dm, \\ &= I_{xx}^C + (A_y^2 + A_z^2)\int_{\mathcal{B}} dm - 2A_y \int_{\mathcal{B}} y dm - 2A_z \int_{\mathcal{B}} z dm. \end{aligned}$$

Because the integral of $\int_{\mathcal{B}} \boldsymbol{\pi} dm = 0$, then $\int_{\mathcal{B}} y dm = 0$ and $\int_{\mathcal{B}} z dm = 0$ and

$$I_{xx}^A = I_{xx}^C + m(A_y^2 + A_z^2).$$

The moment of inertia always increases when we move away from the center of mass.

$$I_{xy}^A = -\int_{\mathcal{B}} (x - A_x)(y - A_y) dm = I_{xy} - mA_xA_y.$$

6 Radius of Gyration

The radius of gyration about the z-axis k_z is a quantity such that

$$mk_z^2 = I_{zz}.$$

k_x and k_y are similarly defined.