MECH230 - Section 2 **DRAFT**
Midterm 2 Formula Sheet

Kinematics in Cartesian Coordinates

$$\begin{aligned}
\mathbf{r} &= x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z, \\
\mathbf{v} &= v_x\mathbf{E}_x + v_y\mathbf{E}_y + v_z\mathbf{E}_z = \dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y + \dot{z}\mathbf{E}_z, \\
\mathbf{a} &= a_x\mathbf{E}_x + a_y\mathbf{E}_y + a_z\mathbf{E}_z = \ddot{x}\mathbf{E}_x + \ddot{y}\mathbf{E}_y + \ddot{z}\mathbf{E}_z.
\end{aligned} \tag{1}$$

Rectilinear Motion Consider a rectilinear motion of a particle in the direction of $\mathbf{E}_x$.

$$\begin{aligned}
\mathbf{r} &= x\mathbf{E}_x, \\
\mathbf{v} &= v\mathbf{E}_x = \dot{x}\mathbf{E}_x, \\
\mathbf{a} &= a\mathbf{E}_x = \ddot{x}\mathbf{E}_x.
\end{aligned} \tag{2}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}. \tag{3}$$

Kinematics in Cylindrical Polar Coordinates

$$\begin{aligned}
\mathbf{r} &= r\mathbf{e}_r + z\mathbf{E}_z, \\
\mathbf{v} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{E}_z, \\
\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\mathbf{E}_z,
\end{aligned} \tag{4}$$

where

$$\mathbf{e}_r = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \qquad \mathbf{e}_\theta = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y. \tag{5}$$

Kinematics in the Serret-Frenet Basis

$$v = ||\mathbf{v}|| = \frac{ds}{dt}, \quad \mathbf{e}_t = \frac{\mathbf{v}}{v}, \quad \frac{d\mathbf{e}_t}{ds} = \kappa\mathbf{e}_n, \quad \mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n, \quad \frac{d\mathbf{e}_b}{ds} = -\tau\mathbf{e}_n, \quad \rho = \frac{1}{\kappa}. \tag{6}$$

$$\begin{aligned}
\mathbf{v} &= v\mathbf{e}_t. \\
\mathbf{a} &= \dot{v}\mathbf{e}_t + \kappa v^2\mathbf{e}_n.
\end{aligned} \tag{7}$$

The Balance of Linear Momentum for a particle $\quad \mathbf{F} = \dot{\mathbf{G}}$ where $\mathbf{G} = m\mathbf{v}$.

Spring Forces A spring of stiffness K with unstretched length ℓ_0 whose base is at point A and whose free end is attached to a mass m with position vector $\mathbf{r}$ applies a force on m that is

$$\mathbf{F}_s = -K(||\mathbf{r} - \mathbf{r}_A|| - \ell_0)\frac{\mathbf{r} - \mathbf{r}_A}{||\mathbf{r} - \mathbf{r}_A||}. \tag{8}$$

Friction Forces

- Static friction is unknown but satisfies that static friction criterion $||\mathbf{F}_f|| \leq \mu_s||\mathbf{N}||$.
- Kinetic friction is prescribed according to Coulomb's friction model to be $\mathbf{F}_f = -\mu_k||\mathbf{N}||\frac{\mathbf{v}_{rel}}{||\mathbf{v}_{rel}||}$.

Power The power of a force $\mathbf{F}$ acting on a particle with velocity $\mathbf{v}$ is defined to be $\mathcal{P} = \mathbf{F} \cdot \mathbf{v}$.

Work The work of a force $\mathbf{F}$ over the interval $[t_A, t_B]$ is $W_{\mathbf{F},AB} = \int_{t_A}^{t_B} \mathbf{F} \cdot \mathbf{v} dt = \int_{\mathbf{r}(t_A)}^{\mathbf{r}(t_B)} \mathbf{F} \cdot d\mathbf{r}$.

Kinetic Energy The kinetic energy of a particle is $T = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v}$.

The work energy theorem for a particle $T_B - T_A = W_{\mathbf{F},AB}$.

A conservative force $\mathbf{F}_c$ is such that $W_{\mathbf{F}_c,AB} = -(U_B - U_A)$. Examples include

- any constant force $\mathbf{C}$ with $U = -\mathbf{C} \cdot \mathbf{r}$,
- the gravitational force $\mathbf{F}_G = G\frac{M_e m}{(R_e+h)^2}(-\mathbf{e}_r)$ with $U = -\frac{GM_e m}{r}$, and
- the spring force $\mathbf{F}_s = -K\varepsilon \frac{\mathbf{r}-\mathbf{r}_A}{||\mathbf{r}-\mathbf{r}_A||}$ with $U = \frac{1}{2}K\varepsilon^2$. The spring stretch $\varepsilon = ||\mathbf{r} - \mathbf{r}_A|| - \ell_0$.

Linear impulse - linear momentum equation $\int_{t_A}^{t_B} \mathbf{F} dt = \mathbf{G}_B - \mathbf{G}_A$.

Angular momentum of a particle about the fixed origin O is $\mathbf{H}^O = \mathbf{r} \times m\mathbf{v}$.

Moment The moment of a force $\mathbf{F}$ applied at point A about point P is $\mathbf{M}^P = (\mathbf{r}_A - \mathbf{r}_P) \times \mathbf{F}$.

Balance of angular momentum of a particle $\mathbf{M}^O = \dot{\mathbf{H}}^O$.

Collisions The pre-impact and post-impact velocities of two particles A and B are related by $e = -\frac{\mathbf{v}'_B \cdot \mathbf{n} - \mathbf{v}'_A \cdot \mathbf{n}}{\mathbf{v}_B \cdot \mathbf{n} - \mathbf{v}_A \cdot \mathbf{n}}$ where e is restitution coefficient and $\mathbf{n}$ is perpendicular to the plane of collision.

Kinematics of a system of particles For a system of n particles each with mass m_i and position vector $\mathbf{r}_i$ from the origin, its center of mass C has

$$\mathbf{r}_C = \frac{\sum_{i=1}^n m_i \mathbf{r}_i}{m = \sum_{i=1}^n m_i}, \quad \mathbf{v}_C = \frac{\sum_{i=1}^n m_i \mathbf{v}_i}{m}, \quad \text{and} \quad \mathbf{a}_C = \frac{\sum_{i=1}^n m_i \mathbf{a}_i}{m}. \tag{9}$$

Its linear momentum is $\mathbf{G} = \sum_{i=1}^n \mathbf{G}_i = m\mathbf{v}_C$.

Its angular momentum about point P is $\mathbf{H}^P = \sum_{i=1}^n (\mathbf{r}_i - \mathbf{r}_P) \times m_i \mathbf{v}_i = \mathbf{H}^C + (\mathbf{r} - \mathbf{r}_P) \times \mathbf{G}$. where $\mathbf{H} = \sum_{i=1}^n (\mathbf{r}_i - \mathbf{r}_C) \times m_k \mathbf{v}_i = \sum_{i=1}^n (\mathbf{r}_i - \mathbf{r}_C) \times m_i(\mathbf{v}_i - \mathbf{v}_C)$.

Its kinetic energy is $T = \sum_{i=1}^n T_i = \sum_{i=1}^n \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \frac{1}{2} m \mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2} \sum_{i=1}^n m_k (\mathbf{v}_i - \mathbf{v}_C) \cdot (\mathbf{v}_i - \mathbf{v}_C)$.

Kinetics of a system of particles

$$\mathbf{F} = \sum_{i=1}^n \mathbf{F}_i = \sum_{i=1}^n m_i \mathbf{a}_i = m\mathbf{a}_c, \quad \text{and} \quad \dot{\mathbf{H}}^P = \mathbf{M}^P - \mathbf{v}^P \times \mathbf{G}. \tag{10}$$

for any point P. If P is a fixed point O, then $\mathbf{M}^O = \dot{\mathbf{H}}^O$ and if P is C, then $\mathbf{M}^C = \dot{\mathbf{H}}^C$.

Its work-energy theorem $\sum_{i=1}^n \dot{T}_i = \sum_{i=1}^n \mathbf{F}_i \cdot \mathbf{v}_i$; $\mathbf{v}_i$ is the velocity of the point where $\mathbf{F}_i$ acts.

Rigid Body (RB) Kinematics If A is fixed to the RB and B moving with respect to it, then

$$\mathbf{r}_B - \mathbf{r}_A = x\mathbf{e}_x + y\mathbf{e}_y, \tag{11}$$
$$\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A) + \mathbf{v}_{rel}, \tag{12}$$
$$\mathbf{a}_B - \mathbf{a}_A = \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)) + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} + \mathbf{a}_{rel}, \tag{13}$$

where the angular velocity and acceleration of the RB are resp. $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ and $\boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z$ and

$$\mathbf{v}_{rel} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y, \quad \text{and} \quad \mathbf{a}_{rel} = \ddot{x}\mathbf{e}_x + \ddot{y}\mathbf{e}_y. \tag{14}$$