Momenta and Impulses

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October 2024

1 Linear Momentum and Its Conservation

Recall that the linear momentum $\mathbf{G}$ of the particle is defined to be

$$\mathbf{G} = m\mathbf{v} = m\dot{r}.$$

1.1 Linear Impulse and Linear Momentum

A more primitive form of the balance of linear momentum $\mathbf{F} = m\mathbf{a}$ is the integral form

$$\mathbf{G}(t_1) - \mathbf{G}(t_0) = \int_{t_0}^{t_1} \mathbf{F} dt.$$

- The time integral of a force is known as its linear impulse.
- This form is more general than $\mathbf{F} = m\mathbf{a}$ because it does not assume that $\mathbf{v}$ can be always differentiated to determine $\mathbf{a}$.

1.2 Conservation of Linear Momentum

Suppose that the component of $\mathbf{G}$ in the direction of a given vector $\mathbf{c}$ is conserved:

$$\frac{d}{dt}(\mathbf{G} \cdot \mathbf{c}) = 0.$$

This means that

$$\dot{\mathbf{G}} \cdot \mathbf{c} + \mathbf{G} \cdot \dot{\mathbf{c}} = \mathbf{F} \cdot \mathbf{c} + \mathbf{G} \cdot \dot{\mathbf{c}} = \mathbf{0}.$$

Thus, given a vector $\mathbf{c}$,

$$\mathbf{G} \cdot \mathbf{c} \text{ is conserved if, and only if, } \mathbf{F} \cdot \mathbf{c} + \mathbf{G} \cdot \dot{\mathbf{c}} = 0.$$

If $\mathbf{c}$ is a constant vector, $\mathbf{G} \cdot \mathbf{c}$ is conserved if, and only if, $\mathbf{F} \cdot \mathbf{c} = 0$. From the BoLM, this means that there is no force in this constant direction.

1.3 Examples

Consider a projectile motion in the $\{\mathbf{E}_x, \mathbf{E}_y\}$ plane with $\mathbf{W} = -mg\mathbf{E}_y$. The linear momentum of the projectile would be conserved in both the $\mathbf{E}_x$ and $\mathbf{E}_y$ directions, but not in the $\mathbf{E}_y$ direction.

2 Angular Momentum and Its Conservation

[ADD FIGURE] Let $\mathbf{r}$ be the position vector of a particle relative to a fixed point O, and let $\mathbf{v}$ be the absolute velocity vector of the particle. Then, the angular velocity momentum of the particle relative to O is denoted by $\mathbf{H}_O$ and defined as

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} = \mathbf{r} \times \mathbf{G}.$$

In Cartesian coordinates,

$$\mathbf{H}_O = \ldots = m\left(y\dot{z} - z\dot{y}\right)\mathbf{E}_x + m\left(z\dot{x} - x\dot{z}\right)\mathbf{E}_y + m\left(x\dot{y} - y\dot{x}\right).$$

In cylindrical-polar coordinates,

$$\mathbf{H}_O = \ldots = -mzr\dot{\theta}\mathbf{e}_r + m\left(z\dot{r} - r\dot{z}\right)\mathbf{E}_y + m\left(x\dot{y} - y\dot{x}\right).$$

2.1 Angular Momentum Theorem

Using the BoLM, we can calculate

$$\dot{\mathbf{H}}_O = \frac{d}{dt}\left(\mathbf{r} \times m\mathbf{v}\right) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}} = \mathbf{r} \times \mathbf{F}.$$

Thus, we obtain the angular momentum theorem

$$\dot{\mathbf{H}}_O = \mathbf{r} \times \mathbf{F}.$$

2.2 Conservation of Angular Momentum

Suppose that the component of $\mathbf{H}_O$ in the direction of a given vector $\mathbf{c}$ is conserved

$$\frac{d}{dt}\left(\mathbf{H}_O \cdot \mathbf{c}\right) = \mathbf{0}.$$

Then,

$$\frac{d}{dt}\left(\mathbf{H}_O \cdot \mathbf{c}\right) = \dot{\mathbf{H}}_O \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}} = \left(\mathbf{r} \times \mathbf{F}\right) \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}}.$$

Consequently, for a given vector $\mathbf{c}$,

$$\mathbf{H}_O \cdot \mathbf{c} \text{ is conserved if, and only if, } \left(\mathbf{r} \times \mathbf{F}\right) \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}} = 0.$$

If $\mathbf{c}$ is a constant vector, then $\mathbf{H}_O \cdot \mathbf{c}$ is conserved if, and only if, $\left(\mathbf{r} \times \mathbf{F}\right) \cdot \mathbf{c} = \dot{\mathbf{H}}_O \cdot \mathbf{c} = 0$.

2.2.1 Central Force Problems

A central force problem is one where $\mathbf{F}$ is parallel to $\mathbf{r}$. The angular momentum theorem in this case implies that $\dot{\mathbf{H}}_O = \mathbf{r} \times \mathbf{F} = \mathbf{0}$.

Then, we can write

$$\mathbf{H}_O = h\mathbf{h} = \mathit{constant} = \mathbf{r} \times m\mathbf{v}$$

where h and $\mathbf{h}$ are constant.

The vectors $\mathbf{r}$ and $\mathbf{v}$ form a plane with a constant unit normal vector $\mathbf{h}$. This plane passes through the origin O and is fixed. Given a set of initial conditions $\mathbf{r}(t_0)$ and $\mathbf{v}(t_0)$, we can choose a cylindrical polar coordinate system such that $\mathbf{E}_z = \mathbf{h}$, $\mathbf{r} = r\mathbf{e}_r$, and $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$. To do this, it suffices to choose $\mathbf{E}_z$ so that

$$\mathbf{H}_O = h\mathbf{E}_z = \mathbf{r}(t_0) \times m\mathbf{v}(t_0).$$

2.3 Kepler's Problem

The most famous example of a central force problem, and angular momentum conservation, was solved by Newton.

Consider the orbit of the Earth about the sun. Recall that the resultant force $\mathbf{F}$ exerted on a planet of mass m by a fixed planet of mass M is conservative:

$$\mathbf{F} = -\frac{GmM}{||\mathbf{r}||^2}\frac{\mathbf{r}}{||\mathbf{r}||} = -\frac{\partial U}{\partial \mathbf{r}}$$
$$U = -\frac{GmM}{||\mathbf{r}||}.$$

Show that the BoLM yields

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{GMm}{r^2},$$
$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = 0.$$

This problem has two conserved quantities.

$$E = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + U = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r}$$
$$h = \mathbf{H}_O \cdot \mathbf{E}_z = mr^2\dot{\theta}$$

Watch this video on Kepler's laws.

2.4 Particle on a Smooth Cone

Show that $\mathbf{H}_O \cdot \mathbf{E}_z$ is conserved.