Power, Work, and Energy

Theresa Honein

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Define the mechanical **power** of a force $\mathbf{P}$ acting on a particle whose absolute velocity is $\mathbf{v}$ as

$$\mathcal{P} = \mathbf{P} \cdot \mathbf{v}.$$

The **work** done by a force $\mathbf{P}$ in an interval of time $[t_A, t_B]$ is the integral of its power with respect to time

$$W_{\mathbf{P},AB} = \int_{t_A}^{t_B} \mathbf{P} \cdot \mathbf{v} dt = \int_{t_A}^{t_b} \mathbf{P} \cdot d\mathbf{r}$$

since

$$d\mathbf{r} = \mathbf{v} dt.$$

So, the power of a force is the derivative of its work $P = \dot{W}$.

In SI units, the **units** of work are Newton meters (or Joule) and the units of power are Newton meters per second (or Watt).

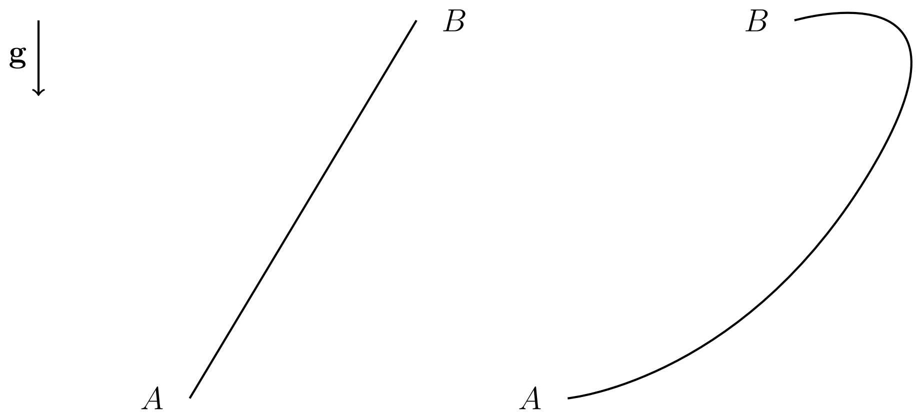
Note.

- Depending on the coordinate system used, there are numerous representations of this integral.
  - In Cartesians, $d\mathbf{r} = dx\mathbf{E}_x + dy\mathbf{E}_y + dz\mathbf{E}_z$.
  - In polar, $d\mathbf{r} = dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta$.
  - In S-F, $d\mathbf{r} = v\mathbf{e}_t dt = s\mathbf{e}_t$.
- This integral is path dependent. (We need to know the particle's path to be able to calculate it.)
- The work of a force is zero if the force is perpendicular to the displacement/velocity.

Example. Particle on smooth rails

Consider two points A and B in the vertical plane connected by a smooth rail. What is the work done by the forces acting on an object as it travels on the rail from B to A if (a) the rail is straight or (b) the rail is a curved path?

In both cases, $\mathbf{F} = \mathbf{W} + \mathbf{N}$. Since $\mathbf{N} \cdot d\mathbf{r} = 0$, the normal force is workless.



The work of the weight is calculated from the definition of work

$$W_{\mathbf{W},1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} mg\mathbf{E}_y \cdot d\mathbf{r} = mg\mathbf{E}_y \cdot \int_{\mathbf{r}_1}^{\mathbf{r}_2} d\mathbf{r} = mg\mathbf{E}_y \cdot \Delta\mathbf{r}_{1-2}.$$

$\Delta\mathbf{r}_{1-2}$ is the same in both rails. $\mathbf{W}$ did the same amount of work.

The work done by $\mathbf{W}$ (constant force) only depends on the endpoints of the path.

Example. Particle on a rough rail

What would change in the previous analysis if the rail is rough?

In this case, $\mathbf{F} = \mathbf{W} + \mathbf{N} + \mathbf{F}_f$.

$$W_{1-2} = W_{\mathbf{W},1-2} + \int_{s_1}^{s_2} \mu_k ||\mathbf{N}||(-\mathbf{e}_t) \cdot \mathbf{e}_t = W_{\mathbf{W},1-2} - \mu_k \int_{s_1}^{s_2} ||\mathbf{N}|| ds.$$

The work of the friction force is path dependent. Also,

- $W_{\mathbf{F}_f,1-2} < 0 \implies$ serves to deplete the kinetic energy
- $W_{\mathbf{W},1-2} \implies$ serves to pump up (increase) the kinetic energy

Example. Particle in elevator.

In this case, the normal force is not workless.

Kinetic Energy

Define the **kinetic energy** of a particle to be

$$T = \frac{m}{2}\mathbf{v} \cdot \mathbf{v}.$$

In the different bases

$$T = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}mv^2.$$

Energy is defined as the ability to perform work.

The Work-Energy Theorem

The rate of change of kinetic energy of a particle is equal to the mechanical power of the resultant force $\mathbf{F}$ acting on the particle.

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v} = \mathcal{P} = \dot{W}.$$

We can prove the work energy theorem for a particle from the balance of linear momentum equation.

$$\frac{d}{dt}T = \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v} \cdot \mathbf{v}\right) = m\mathbf{a} \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v}.$$

You might be familiar with the integral form of this theorem (integral with respect to time).

$$T_B - T_A = W_{\mathbf{F},AB} = \int_{t_A}^{t_B} \mathbf{F} \cdot d\mathbf{r}.$$

<u>Implications of $\dot{T} = \mathbf{F} \cdot \mathbf{v}$ through examples</u> [Include sketches.]

- Consider a free falling particle. The weight is the only force acting on it. If the particle is going up (opposite to the weight), $\dot{T} < 0$. If the particle is going down (same direction of the weight), $\dot{T} > 0$.
- Consider a person standing on a train:

  $$\begin{aligned} \mathbf{F} &= \mathbf{W} + \mathbf{F}_f + \mathbf{N} \\ \mathbf{F} \cdot \mathbf{v} &= \mathbf{F}_f \cdot \mathbf{v} \geq 0 \implies \dot{T} > 0. \end{aligned}$$

  The kinetic energy of the person is increasing.
- Consider a box sliding on a rough surface.

  $$\begin{aligned} \mathbf{F} &= \mathbf{F}_f + \mathbf{W} + \mathbf{N} \\ \mathbf{F} \cdot \mathbf{v} &= \mathbf{F}_f \cdot \mathbf{v} < 0 \implies \dot{T} < 0. \end{aligned}$$

  The kinetic energy of the box is decreasing until it stops.

- Consider a particle on a smooth rail in the horizontal plane.

$$\begin{aligned}\mathbf{F} &= \mathbf{W} + \mathbf{N}\\ \mathbf{F}\cdot\mathbf{v} &= 0 \implies \dot{T} = 0.\end{aligned}$$

  The kinetic energy of the particle is conserved.

You are probably more familiar with the integral form of this equation

$$W_{1-2} = T_2 - T_1.$$

Examples from the problem set.

Conservative forces

Forces whose work done depends only on the endpoints of a path are conservative forces.

From the previous examples, we deduce that weight $\mathbf{W}$ (or constant forces) are conservative. Friction forces are not conservative.

A force $\mathbf{F}_c$ is conservative if one can find a scalar function (called a potential energy function) $U = U(\mathbf{r})$ from which $\mathbf{F}_c$ is derivable:

$$\mathbf{F}_c = -\frac{\partial U}{\partial \mathbf{r}} = -\text{grad}_{\mathbf{r}} U.$$

The minus sign in conventional.

Check:

$$W_{1-2} = \int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \mathbf{F}_c \cdot d\mathbf{r} = -\int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \frac{\partial U}{\partial \mathbf{r}} \cdot d\mathbf{r} = -U_2 + U_1 = -\Delta_{1-2} U.$$

U_2 and U_1 are the potential energies evaluated at the endpoints.

If the potential energy of a force is decreased ($U_2 \leq U_1$), $W_{1-2} > 0$ and the kinetic energy is increased.

<u>Example: Particle in free fall</u>

Consider a particle thrown from the top of a building. When the particle is still high, its potential energy is high and its kinetic energy is low. When the particle has traveled downward, its potential energy is low and its kinetic energy is high.

Constant Forces Constant force $\mathbf{C}$ are conservative.

Proof: Propose $\mathbf{U} = -\mathbf{C} \cdot \mathbf{r} = U(\mathbf{r}) = -C_x x - C_y y - C_z z$.

$$-\text{grad}_r U = C_x \mathbf{E}_x + C_y \mathbf{E}_y + C_z \mathbf{E}_z = \mathbf{C}.$$

We found a U from which $\mathbf{C}$ is derivable.

The weight near the Earth's surface is a constant force

$$U_{\mathbf{W}} = -[-mg\mathbf{E}_y] \cdot \mathbf{r} = mgy.$$

Spring Force The spring force is also a conservative force. Recall the spring force

$$\mathbf{F}_s = K\varepsilon \left(-\frac{\mathbf{r} - \mathbf{r}_A}{||\mathbf{r} - \mathbf{r}_A||} \right)$$
$$\epsilon = ||\mathbf{r} - \mathbf{r}_A|| - \ell_0$$

where ϵ is the stretch.

Suppose the spring potential energy is

$$U_s = \frac{1}{2} K\varepsilon.$$

$U_s > 0$ whether spring is in compression or tension.

An increase in potential energy denotes an increase in the capacity of a spring to do mechanical work on its surroundings in either case.

We can show that

$$-\frac{\partial U_s}{\partial \mathbf{r}} = K\varepsilon \frac{\mathbf{r}_A - \mathbf{r}}{||\mathbf{r}_A - \mathbf{r}||} = \mathbf{F}_s.$$

$\implies$ $\mathbf{F}_s$ is conservative.

Drag forces, friction forces, tension forces in inextensible cables, and normal forces are nonconservative.

Energy and its conservation.

Why are $\mathbf{F}_c$ called conservative?

If only conservative forces act on a system, we can find a conserved quantity:

$$\begin{aligned}\dot{T} &= \mathbf{F}_c \cdot \mathbf{V} = -\frac{\partial U}{\partial \mathbf{r}} \cdot \dot{\mathbf{r}} = -\dot{U} \\ \dot{T} &= -\dot{U} = \dot{\overline{T+U}} = 0.\end{aligned}$$

$T+U$ is conserved, call it $E = T_U$, the total mechanical engergy.

If $\mathbf{F} \cdot \mathbf{v} = \mathbf{F}_c \cdot \mathbf{v}$, then $\dot{E} = 0$. In other words, if all the forces doing work on the system are conservative, then the total energy of the system is conserved.

So, we have introduced 3 types of energies

- kinetic energy T
- Potential energy U
- Mechanical energy E

What is energy? It is the particle's capacity to perform mechanical work on its surrounding.

If energy is not conserved, then the change in energy is equal to the work of the nonconservative forces $\mathbf{F}_{nc}$.

$$E_B - E_A = W_{\mathbf{F}_{nc},AB}. \tag{1}$$

<u>Example: understanding energy</u>

- A static box has not much energy.
- A morning box has more energy. It could collide with another box and do work on it (change its velocity).

<u>Example: smooth rail vs. rough rail</u>

Consider a particle moving on (a) a smooth rail (b) a rough rail.

In (a)

$$\begin{aligned}\mathbf{F} &= \mathbf{W} + \mathbf{N} \\ \dot{T} &= \mathbf{W} \cdot \mathbf{v} + \mathbf{N} \cdot \mathbf{v} = \mathbf{W} \cdot \mathbf{v}. \\ \dot{T} &= -\dot{U}_{\mathbf{W}} \\ \dot{E} &= 0\end{aligned}$$

$\implies E$ is a conserved quantity.

In (b)

$$\begin{aligned}\mathbf{F} &= \mathbf{W} + \mathbf{N} + \mathbf{F}_f \\ \dot{T} &= \mathbf{W} \cdot \mathbf{v} + \mathbf{F}_f \cdot \mathbf{v} \\ \dot{T} + \dot{U}_{grav} &= \mathbf{F}_f \cdot \mathbf{v} < 0 \\ \dot{E} &< 0\end{aligned}$$

$\implies E$ is bleeding off in time.

In the case with no friction (a), we can exploit the conservation of energy to solve problems. For example, we can evaluate the energy at two finitely separated time instances t_1 and t_2: $E(t_1) = E(t_2)$.

In the case with friction (b),

$$\begin{aligned}
&\dot{E} = \mathbf{F}_f \cdot \mathbf{v} \\
&\int_{t_1}^{t^2} \dot{E} dt = \int_{t_1}^{t_2} \mathbf{F}_f \cdot \mathbf{v} dt \\
&E(t_2) - E(t_1) = W_{\mathbf{F}_f, 1-2} \\
&E(t_1) + W_{\mathbf{F}_f, 1-2} = E(t_2).
\end{aligned}$$

Since $W_{\mathbf{F}_f, 1-2} < 0$, $E(t_2) < E(t_1)$.