Example: Position, Velocity, Acceleration, and Cross Product

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Consider a particle p moving in $\mathbb{E}^2$ with position vector

$$\mathbf{r} = R_0 \left(\cos\left(\omega t\right) \mathbf{E}_x + \sin\left(\omega t\right) \mathbf{E}_y\right).$$

We calculate its velocity and position vectors as follows:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = R_0\omega \left(-\sin\left(\omega t\right) \mathbf{E}_x + \cos\left(\omega t\right) \mathbf{E}_y\right),$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -R_0\omega^2 \left(\cos\left(\omega t\right) \mathbf{E}_x + \sin\left(\omega t\right) \mathbf{E}_y\right).$$

As you take these derivatives, remember that

- $\dot{\mathbf{E}}_x = \mathbf{0}$ and $\dot{\mathbf{E}}_y = \mathbf{0}$ since the basis $\{\mathbf{E}_x, \mathbf{E}_y\}$ is fixed, and
- the chain rule $(f(g(x)))' = f'(g(x))g'(x)$.

Then,

$$\begin{aligned}
&\mathbf{r} \times \mathbf{a} \\
&= R_0 \left(\cos\left(\omega t\right) \mathbf{E}_x + \sin\left(\omega t\right) \mathbf{E}_y\right) \times \left(-R_0\omega^2 \left(\cos\left(\omega t\right) \mathbf{E}_x + \sin\left(\omega t\right) \mathbf{E}_y\right)\right), \\
&= -R_0^2\omega^2 \left(\cos^2(\omega t)\, \underbrace{\mathbf{E}_x \times \mathbf{E}_x}_{\mathbf{0}} + \cos(\omega t)\sin(\omega t) \left(\underbrace{\mathbf{E}_x \times \mathbf{E}_y}_{\mathbf{E}_z} + \underbrace{\mathbf{E}_y \times \mathbf{E}_x}_{-\mathbf{E}_z}\right) + \sin(\omega t)^2\, \underbrace{\mathbf{E}_y \times \mathbf{E}_y}_{\mathbf{0}}\right), \\
&= \mathbf{0}.
\end{aligned}$$

Alternatively, one can also notice that

$$\mathbf{a} = -\omega^2\mathbf{r},$$

so

$$\mathbf{r} \times \mathbf{a} = \mathbf{r} \times -\omega^2 \mathbf{r} = -\omega^2 \mathbf{r} \times \mathbf{r} = \mathbf{0}. \tag{1}$$

Let's for practice also calculate $\mathbf{r} \cdot \mathbf{a}$.

$$\begin{aligned}
&\mathbf{r} \cdot \mathbf{a} \\
&= R_0 \left(\cos\left(\omega t\right) \mathbf{E}_x + \sin\left(\omega t\right) \mathbf{E}_y\right) \cdot \left(-R_0 \omega^2 \left(\cos\left(\omega t\right) \mathbf{E}_x + \sin\left(\omega t\right) \mathbf{E}_y\right)\right), \\
&= -R_0^2 \omega^2 \left(\cos^2(\omega t) \underbrace{\mathbf{E}_x \cdot \mathbf{E}_x}_{1} + \cos(\omega t) \sin(\omega t) \left(\underbrace{\mathbf{E}_x \cdot \mathbf{E}_y}_{0} + \underbrace{\mathbf{E}_y \cdot \mathbf{E}_x}_{0} \right) + \sin(\omega t)^2 \underbrace{\mathbf{E}_y \cdot \mathbf{E}_y}_{1} \right) \\
&= -R_0^2 \omega^2 \left(\cos^2(\omega t) + \sin^2(\omega t)\right) \\
&= -R_0^2 \omega^2.
\end{aligned} \tag{2}$$

Alternatively,

$$\mathbf{r} \cdot \mathbf{a} = \mathbf{r} \cdot \left(-\omega^2 \mathbf{r}\right) = -\omega^2 \mathbf{r} \cdot \mathbf{r} = \omega^2 \left\|\mathbf{r}\right\| = -\omega^2 R_0^2.$$