On the relationship between holonomy and nonintegrable constraints

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October 5, 2024

A nonintegrable constraint f written in terms of position coordinates q and velocity coordinates u cannot be expressed as the derivative of some function F of position coordinates only. This statement can be summarized as

$$f(t,q,u) = 0 \neq \frac{dF(t,q)}{dt}. \tag{1}$$

While an integrable constraint reduces the degrees of freedom of a system, or equivalently, reduces the dimension of a configuration manifold, a nonintegrable constraint does not. A nonintegrable constraint restricts the path of the superpaticle in the configuration manifold without reducing the manifold's dimension. We can visualized a two dimensional configuration manifold with the coordinates $\{u_1, u_2\}$ and curvilinear coordinate lines as follows.

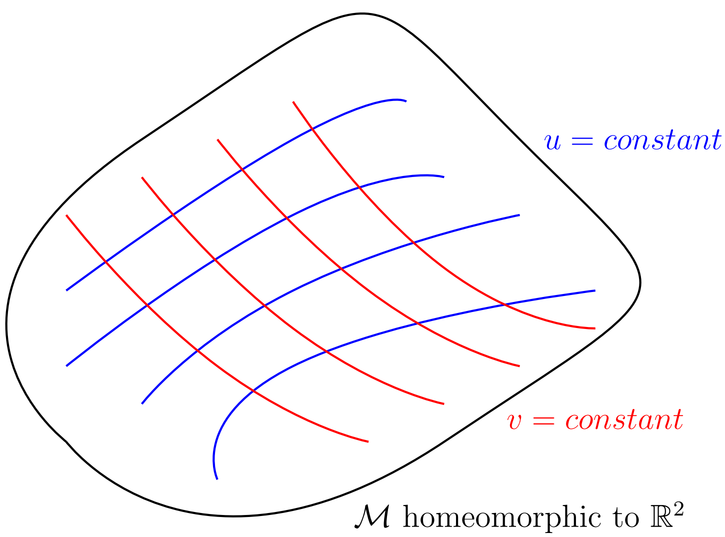

Figure 1: A sketch of a two dimensional manifold with the u and v curvilinear coordinate lines are blue and read respectively.

If the system is not subject to nonintegrable constraints, $\{u, v\}$ can change independently. However, is the system is subject to integrable constraints, then $\{u, v\}$ cannot change independently.

Consider a cylinder subject to two integrable constraints: the cylinder is in point contact with a horizontal plane and the cylinder is held such that it maintains a constant inclination angle with the vertical.

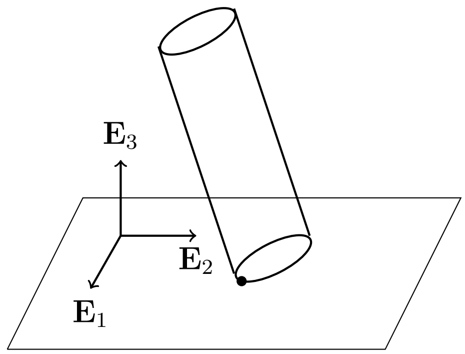

Figure 2: A cylinder in point contact with a horizontal plane. The cylinder maintains a constant inclination angle with the vertical.

An unconstrained cylinder, like any rigid body, has 6 degrees of freedom, and its configuration space is a 6-dimensional manifold. These degrees of freedom can be parametrized by the coordinates of its mass center $\{x_i\}, i = 1, 2, 3$ and by the 3-1-3 Euler angles $\{\psi, \theta, \phi\}$ describing its orientation. Now a cylinder constrained as previously described has a reduced configurations space that is a 4-dimensional manifold. The integrable constraints determine x_3 and θ, so the remaining degrees of freedom are $\{x_1, x_2, \psi, \phi\}$.

If the cylinder is not subject to a nonintegrable constraint such as the roll without slip constraints, all the coordinates of the cylinder can be changed independently with changing the rest: x_1 and x_2 can be changed simply by sliding the cylinder parallel to the coordinate axes, ψ can be changed by rotating the cylinder about $\mathbf{E}_3$, and ϕ can be changed by rotating the cylinder about its axis.

If the cylinder is subject to the roll without slip nonintegrable constraints, than these 4 remaining degrees of freedom cannot be changes independently. Furthermore, consider the case when the cylinder traverses a closed path such that $\{x_1, x_2, \psi\}$ return to their original values. The angle ϕ does not necessarily return to its original value. The change in ϕ would depend on the path traversed by the particle.

This phenomenon is known as holonomy. Holonomy can be defined as the change in some position coordinates of a system while the rest have returned to their original values.

It is well known that the phenomenon of holonomy is a consequence of the action of one or more nonintegrable constraints on the dynamical system. We how just illustrated this relationship by way of example.

This whole thing somehow reminds me of conservative and nonconservative forces. Conservative forces can be written as a gradient of some potential energy function, and their work is path independent, only depending on the endpoints of a system. So if the system traces a closed path, the work of a conservative force during that motion would be zero. On the other hand, the work on a nonconservative force would be nonzero, and would actually depend on the path traversed by the system. In this scenario, constraints would be analogous to the work differential of a constraint $\mathbf{F}$ acting on a particle with position vector $\mathbf{r}$, $dW = \mathbf{F} \cdot d\mathbf{r}$.