On a basis formed by the quaternion representation of a rotation

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Consider a rigid body rotating in $\mathbb{E}^3$. Let $\{\mathbf{E}^1, \mathbf{E}^2, \mathbf{E}^3\}$ be a fixed basis for the space and $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ be a corotational basis attached to the rigid body.

Leonard Euler described the rotation of a rigid body using an angle of rotation ϕ and a unit axis of rotation $\mathbf{r}$ ($\mathbf{r} \cdot \mathbf{r} = 1$) [1]. In terms of tensors,

$$\mathbf{R} = \mathbf{L}(\phi, \mathbf{r}) = \cos(\phi)(\mathbf{I} - \mathbf{r} \otimes \mathbf{r}) + \sin(\phi)\mathrm{skwt}(\mathbf{r}) + \mathbf{r} \otimes \mathbf{r}. \tag{1}$$

This rotation tensor is written in terms of three parameters, ϕ and the components of the axis of rotation $\mathbf{r} = r_1\mathbf{E}_1 + r_2\mathbf{E}_2 + \sqrt{1 - r_1^2 - r_2^2}\mathbf{E}_3$.

The 12 Euler angle sets are also a three parameter representation of rotations,

$$\mathbf{R} = \mathbf{L}(\nu^3, \mathbf{g}_3)\mathbf{L}(\nu^2, \mathbf{g}_2)\mathbf{L}(\nu^1, \mathbf{g}_1) \tag{2}$$

where $\{\nu^1, \nu^2, \nu^3\}$ are the Euler angles and $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ is the Euler basis which is written in terms of the Euler angles.

Every set of Euler angles has a singularity,

$$\begin{array}{ll} 121, 131, 212, 232, 313, 323, & \text{singularity at } \nu^2 = k\pi, k \in \mathbb{Z} \\ 123, 132, 213, 231, 321, 312, & \text{singularity at } \nu^2 = \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \end{array} \tag{3}$$

At the singularity, $[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = 0$, In other words, the vectors $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ are not linearly independent, and do not form a basis for $\mathbb{E}^3$.

The quaternion representation of a rotation is[1]

$$\mathbf{p} = q_0 + \mathbf{q}. \tag{4}$$

where the scalar q_0 and the vector $\mathbf{q}$ are respectively

$$\begin{aligned} q_0 &= \cos\left(\frac{\phi}{2}\right), \\ \mathbf{q} &= \sin\left(\frac{\phi}{2}\right)(r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3). \end{aligned} \tag{5}$$

Consider the basis $\{\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q} \times \dot{\mathbf{q}}\}$, which we call the *quaternion basis*. Does this basis have any singularities?

To make this determination, we calculate

$$\begin{aligned} \mathbf{q} \times \dot{\mathbf{q}} &= \sin\left(\frac{\phi}{2}\right)\mathbf{r} \times \left(\frac{\dot{\phi}}{2}\cos\left(\frac{\phi}{2}\right)\mathbf{r} + \sin\left(\frac{\phi}{2}\right)\dot{\mathbf{r}}\right), \\ &= \sin^2\left(\frac{\phi}{2}\right)\mathbf{r} \times \dot{\mathbf{r}}. \end{aligned} \tag{6}$$

Since $\mathbf{r}$ is a unit vector and $\mathbf{r} \cdot \dot{\mathbf{r}} = 0$, $\mathbf{r}$ and $\dot{\mathbf{r}}$ are perpendicular, we conclude that for any rotation through an angle $\phi \in (0, 2\pi)$, $\mathbf{q} \times \dot{\mathbf{q}} \neq 0$. Thus, the quaternion basis is free of singularities.

Finally, we note that the angular velocity vector in terms of the different basis can be written as

$$\begin{aligned} \boldsymbol{\omega} &= \mathrm{skew}(\dot{\mathbf{R}}\mathbf{R}^T), \\ &= \dot{\nu}^1\mathbf{g}_1 + \dot{\nu}^2\mathbf{g}_2 + \dot{\nu}^3\mathbf{g}_3, \\ &= 2(q_0\dot{\mathbf{q}} - \dot{q}_0\mathbf{q} + \mathbf{q} \times \dot{\mathbf{q}}). \end{aligned} \tag{7}$$

The latter representation of the angular velocity vector inspires the quaternion basis.

[1] The axis of rotation has the same components on the fixed and on the corotational basis.

References

[1] Leonhard Euler. "Nova methodus motum corporum rigidorum degerminandi". In: *Novi commentarii academiae scientiarum Petropolitanae* (1776), pp. 208–238. URL: http://eulerarchive.maa.org/pages/E479.html.